Cambridge Assessment International Education
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/13**

Paper 1 **May/June 2019**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2019 series for most Cambridge IGCSE™, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:

Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:

Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:

Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:

Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:

Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:

Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

Abbreviations

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no “follow through” from a previous error is allowed)

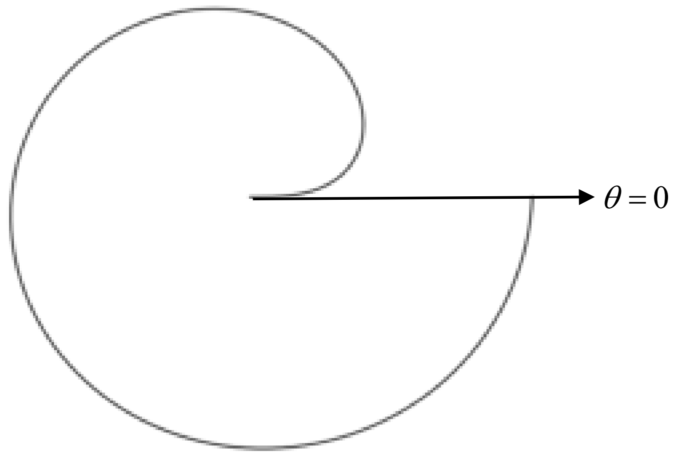
CWO Correct Working Only – often written by a ‘fortuitous’ answer

ISW Ignore Subsequent Working

SOI Seen or implied

SC Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Question	Answer	Marks	Guidance
1	$3^3 - 1 = 26$ is divisible by 13	B1	Checks base case.
	Assume that $3^{3k} - 1$ is divisible by 13 for some positive integer k	B1	States inductive hypothesis.
	Then $3^{3k+3} - 1 = 3^3 3^{3k} - 1 = 26 \cdot 3^{3k} + 3^{3k} - 1$	M1	Separates $3^{3k} - 1$
	is divisible by 13	A1	
	$H_k \Rightarrow H_{k+1}$ By induction, $3^{3n} - 1$ is divisible by 13 for every positive integer n.	A1	States conclusion.
		5	

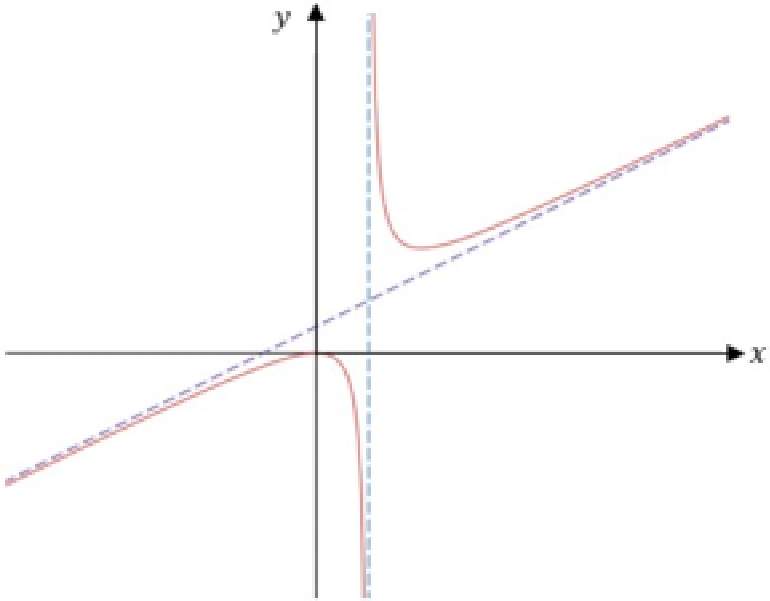
Question	Answer	Marks	Guidance
2(i)		B1	Correct shape, domain and orientation.
		B1	The initial line is tangential to C at the pole.
		2	
2(ii)	$A = \frac{1}{2}\int_0^{2\pi} \ln(1+\theta)\,d\theta$	M1	States $\frac{1}{2}\int r^2\,d\theta$ with correct expression and limits.
	$\int_0^{2\pi} \ln(1+\theta)\,d\theta = \int_1^{2\pi+1} \ln u\ du$	M1	Applies given substitution correctly, changes their limits.
	$\int_1^{2\pi+1} \ln u\ du = [u \ln u]_1^{2\pi+1} - [u]_1^{2\pi+1}$	M1 A1	Integrates $\ln u$ (by parts or otherwise).
	$A = \left(\pi + \frac{1}{2}\right)\ln(2\pi+1) - \pi$	A1	AEF, must be exact.
		5	

Question	Answer	Marks	Guidance
3(i)	$\exp\left(\mathrm{i}\frac{2\pi k}{5}\right),\ k=0,\pm1,\pm2$	B2	B1 for 1 correct fifth root. B1 for all 5 distinct, correct roots (AEF). SCB1 if only arguments are given.
3(ii)	$z^5=-\frac{1}{2}\pm\mathrm{i}\frac{\sqrt{3}}{2}$	M1	Correctly solves quadratic in z^5
	$z^5=\exp\left(\mathrm{i}2\pi\left(\frac{1}{3}+k\right)\right)$ or $\exp\left(\mathrm{i}2\pi\left(-\frac{1}{3}+k\right)\right)$	M1 A1	Writes in polar or exponential form and adds multiples of 2π
	$\exp\left(\pm\mathrm{i}\frac{2\pi}{15}\right),\ \exp\left(\pm\mathrm{i}\frac{4\pi}{15}\right),\exp\left(\pm\mathrm{i}\frac{8\pi}{15}\right),\exp\left(\pm\mathrm{i}\frac{2\pi}{3}\right),\exp\left(\pm\mathrm{i}\frac{14\pi}{15}\right)$	A2	A1 for 5 distinct, correct roots. A1 for exactly 10 distinct, correct roots. Allow alternative exact values of θ such as $\theta=\frac{16\pi}{15},\frac{4\pi}{3},\frac{22\pi}{15},\frac{26\pi}{15},\frac{28\pi}{15}$.
	Alternative method for 3(ii)		
	$z^5+z^{-5}=-1$	M1	Divides through by z^5
	$2\cos 5\theta=-1$	M1 A1	Applies de Moivre's theorem
	$\exp\left(\pm\mathrm{i}\frac{2\pi}{15}\right),\ \exp\left(\pm\mathrm{i}\frac{4\pi}{15}\right),\exp\left(\pm\mathrm{i}\frac{8\pi}{15}\right),\exp\left(\pm\mathrm{i}\frac{2\pi}{3}\right),\exp\left(\pm\mathrm{i}\frac{14\pi}{15}\right)$	A2	A1 for 5 distinct, correct roots. A1 for exactly 10 distinct, correct roots. Allow alternative exact values of θ such as $\theta=\frac{16\pi}{15},\frac{4\pi}{3},\frac{22\pi}{15},\frac{26\pi}{15},\frac{28\pi}{15}$.
		5	

Question	Answer	Marks	Guidance
4(i)	$\frac{1}{(3r+1)(3r-2)} = \frac{1}{3}\left(\frac{1}{3r-2} - \frac{1}{3r+1}\right)$	M1 A1	Finds partial fractions.
	$\sum_{r=1}^{N} \frac{1}{(3r+1)(3r-2)} = \frac{1}{3}\left(\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \ldots \frac{1}{3N-2} - \frac{1}{3N+1}\right)$	M1	At least 3 term including final term.
	$= \frac{1}{3}\left(1 - \frac{1}{3N+1}\right)$	A1	AG
		4	
4(ii)	$\sum_{r=N+1}^{N^2} \frac{N}{(3r+1)(3r-2)} = \sum_{r=1}^{N^2} \frac{N}{(3r+1)(3r-2)} - \sum_{r=1}^{N} \frac{N}{(3r+1)(3r-2)}$	M1	Uses $\sum_{r=N+1}^{N^2} = \sum_{r=1}^{N^2} - \sum_{r=1}^{N}$
	$= \frac{N}{3} - \frac{N}{3(3N^2+1)} - \left(\frac{N}{3} - \frac{N}{3(3N+1)}\right)$	M1	Applies **(i)**
	$= \frac{N}{3(3N+1)} - \frac{N}{3(3N^2+1)} = \frac{N^3 - N^2}{(3N+1)(3N^2+1)}$	A1	Allow simplification to common denominator.
	$\to \frac{1}{9}$ as $N \to \infty$	B1	
		4	

Question	Answer	Marks	Guidance
5(i)	$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 5 & 2 & 1 & -3 \\ 4 & 0 & 1 & -7 \\ -2 & 4 & -1 & \alpha \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -8 & 1 & -23 \\ 0 & -8 & 1 & -23 \\ 0 & 8 & -1 & \alpha+8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -8 & 1 & -23 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha-15 \end{pmatrix}$	M1 A1	Performs row operations correctly.
	$r(\mathbf{M}) = 2 \Rightarrow \alpha = 15$	A1	CWO
	Basis for the range space is $\left\{\begin{pmatrix} 1 \\ 5 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 4 \end{pmatrix}\right\}$	B1	Accept any two independent column vectors from **M**.
		4	
5(ii)	$x + 2y \quad + 4t = 0$ $-8y + z - 23t = 0$	M1	Forms system of equations.
	$t = \lambda,\ z = \mu,\ y = \frac{1}{8}\mu - \frac{23}{8}\lambda,\ x = -\frac{1}{4}\mu + \frac{7}{4}\lambda$	M1	Uses two parameters.
	Basis for null space is $\left\{\begin{pmatrix} -2 \\ 1 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 14 \\ -23 \\ 0 \\ 8 \end{pmatrix}\right\}$	A1 A1	AEF. Many alternatives are possible e.g. $\left\{\begin{pmatrix} -2 \\ 1 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 23 \\ 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 4 \\ 0 \\ -23 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -7 \\ -1 \end{pmatrix}\right\}$
		4	

Question	Answer	Marks	Guidance
6(i)	$x=\frac{1}{k}$	B1	States vertical asymptote.
	$y=\frac{(kx-1)\left(k^{-1}x+k^{-2}\right)+k^{-2}}{kx-1}$	M1	Finds oblique asymptote.
	Oblique asymptote is $y=k^{-1}x+k^{-2}$	A1	
		3	
6(ii)	$y'=\frac{2x(kx-1)-kx^2}{(kx-1)^2}=0\Rightarrow kx^2-2x=0$	M1	Differentiates and equates to 0.
	$x=0,\ 2k^{-1}$	A1	Finds x-coordinates.
	$(0,0),\ \left(2k^{-1},4k^{-2}\right)$	A1	Finds y-coordinates
		3	

Question	Answer	Marks	Guidance
6(iii)		B1	Axes and asymptotes correct.
		B1	Upper branch correct.
		B1	Lower branch correct. Deduct at most 1 mark for poor forms at infinity.
		3	

Question	Answer	Marks	Guidance
7(i)	$\overrightarrow{AB}=\begin{pmatrix}2\\4\\5\end{pmatrix},\ \overrightarrow{CD}=\begin{pmatrix}1\\1\\0\end{pmatrix}$	B1	Find the directions of the lines.
	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&4&5\\1&1&0\end{vmatrix}=\begin{pmatrix}-5\\5\\-2\end{pmatrix}$	M1 A1	Finds direction of common perpendicular. Allow any non-zero scalar multiple.
	$\dfrac{\begin{pmatrix}1\\5\\1\end{pmatrix}\cdot\begin{pmatrix}-5\\5\\-2\end{pmatrix}}{\sqrt{5^2+5^2+2^2}}=\dfrac{18}{\sqrt{54}}=\sqrt{6}=2.45$	M1 A1	Uses formula for shortest distance.
		5	
7(ii)	$\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&4&5\\2&6&1\end{vmatrix}=\begin{pmatrix}-26\\8\\4\end{pmatrix}\sim\begin{pmatrix}-13\\4\\2\end{pmatrix}$	M1 A1	Finds normal to the plane.
	$\dfrac{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-13\\4\\2\end{pmatrix}}{\sqrt{1^2+1^2+0^2}\sqrt{13^2+4^2+2^2}}=-\dfrac{9}{\sqrt{378}}=-\dfrac{\sqrt{42}}{14}$	M1 A1	**FT** Uses dot product of their normal with direction of line.
	$\cos^{-1}\left(-\dfrac{\sqrt{42}}{14}\right)-90=27.6°\quad(\text{or } 0.481\text{ rad})$	A1	
		5	

Question	Answer	Marks	Guidance
8	$9u^2+6u+1=0 \Rightarrow (3u+1)^2=0$	M1	Auxiliary equation.
	CF: $x=(At+B)\mathrm{e}^{-\frac{1}{3}t}$	A1	States CF.
	$x=p\sin t+q\cos t \Rightarrow \dot{x}=p\cos t-q\sin t$ $\Rightarrow \ddot{x}=-p\sin t-q\cos t$	M1	Forms PI and differentiates.
	$-9p-6q+p=50$ $-9q+6p+q=0$	M1	Substitutes.
	$q=-3, p=-4.$	A1	
	$x=(At+B)\mathrm{e}^{-\frac{1}{3}t}-4\sin t-3\cos t$	A1	States general solution. FT if correct form of CF/PI.
	$\dot{x}=-\frac{1}{3}(At+B)\mathrm{e}^{-\frac{1}{3}t}+A\mathrm{e}^{-\frac{1}{3}t}-4\cos t+3\sin t$	M1	Differentiates and forms equations using initial conditions.
	$B=3$	B1	
	$-\frac{1}{3}B+A-4=0 \Rightarrow A=5$	A1	.
	$x=(5t+3)\mathrm{e}^{-\frac{1}{3}t}-4\sin t-3\cos t$	A1	States PS.
		10	

Question	Answer	Marks	Guidance
9(i)	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma} \Rightarrow c=\alpha\beta+\beta\gamma+\alpha\gamma=5$	M1 A1	Uses $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$.
	$d=12$	B1	
		3	
9(ii)	$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	M1	
	$(\alpha+\beta+\gamma)^2-10=b^2-10$	A1	
	Alternative method for 9(ii)		
	$90+bS_2-5b+36=0 \Rightarrow S_2=\frac{5b-126}{b}$	M1 A1	Sums over roots
		2	

Question	Answer	Marks	Guidance
9(iii)	$x^3 + bx^2 + 5x + 12 = 0$ $\quad (-b = \alpha + \beta + \gamma)$	M1	Formulates equation.
	$90 + b\left(b^2 - 10\right) - 5b + 36 = 0$	M1 A1	Sums over roots and uses **9(ii)**
	$b^3 - 15b + 126 = 0$	A1	
	Alternative method 1 for 9(iii)		
	$(\alpha+\beta+\gamma)^3 = \alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma$	M1 A1	Uses expansion of $(\alpha+\beta+\gamma)^3$
	$-b^3 = 90 - 15b + 36$	(M1)	Substitutes known values.
	$b^3 - 15b + 126 = 0$	A1	
	Alternative method 2 for 9(iii)		
	$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = b^2 - 10$	M1 A1	Finds sum of squares in terms of b.
	$\dfrac{5b-126}{b} = b^2 - 10 \Rightarrow b^3 - 15b + 126 = 0$	M1 A1	Equates expressions.
		4	
9(iv)	Real root is $b = -6$	M1 A1	Gives the real root for b.

Question	Answer	Marks	Guidance
10(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{n+1}x\right)=-(n+1)\cot^{n}x\csc^{2}x$	M1 A1	Differentiates using chain rule.
	$=-(n+1)\cot^{n}x\left(\cot^{2}x+1\right)$ $=-(n+1)\cot^{n+2}x-(n+1)\cot^{n}x$	M1	Uses $\csc^{2}x=\cot^{2}x+1$. OE
	$\left[\cot^{n+1}x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}=-(n+1)\left(I_{n+2}+I_{n}\right)$	M1	Integrates both sides.
	$-1=-(n+1)\left(I_{n+2}+I_{n}\right)\Rightarrow I_{n+2}=\frac{1}{n+1}-I_{n}$	A1	AG
10(i)	**Alternative method for 10(i)**		
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos^{n+1}x}{\sin^{n+1}x}\right)=\frac{-(n+1)\sin^{n+2}x\cos^{n}x-(n+1)\sin^{n}x\cos^{n+2}x}{\sin^{2n+2}x}$	M1 A1	Differentiates using quotient/chain rule.
	$-(n+1)\cot^{n}x-(n+1)\cot^{n+2}x$	M1	Separates
	$\left[\cot^{n+1}x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}=-(n+1)\left(I_{n+2}+I_{n}\right)$	M1	Integrates both sides.
	$-1=-(n+1)\left(I_{n+2}+I_{n}\right)\Rightarrow I_{n+2}=\frac{1}{n+1}-I_{n}$	A1	AG
		5	

Question	Answer	Marks	Guidance
10(ii)	$\bar{y} = \dfrac{1}{2A}\displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^2 x \, dx = \dfrac{I_2}{2A}$	M1	Uses correct formula for $\bar{y}$. Allow incorrect or missing limits.
	$A = \displaystyle\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot x \, dx = \left[\ln \sin x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = \ln\sqrt{2}$	M1 A1	Integrates $\cot x$ to find area under the curve with correct limits.
	$I_2 = 1 - I_0 = 1 - \dfrac{1}{4}\pi$	M1 A1	Finds I_2
	$\bar{y} = \dfrac{1}{\ln 2}\left(1 - \dfrac{1}{4}\pi\right)$	A1	AEF. Must be exact.
		6	

Question	Answer	Marks	Guidance
11E(i)	$\mathbf{P}=\begin{pmatrix}1&-1&0\\1&0&1\\0&b&-1\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}2&0&0\\0&1&0\\0&0&3\end{pmatrix}$	B1 B1	Writes **P** and **D** (accept correctly matched permutations of columns)
	$\det\mathbf{P}=-b-1$	B1	
	$\mathbf{P}^{-1}=\frac{1}{b+1}\begin{pmatrix}b&-1&-b\\1&1&b\\1&1&-1\end{pmatrix}^{\mathrm{T}}=\frac{1}{b+1}\begin{pmatrix}b&1&1\\-1&1&1\\-b&b&-1\end{pmatrix}$	M1 A1	Finds inverse of **P**. (Adj ÷ Det).
	$\mathbf{A}=\mathbf{PDP}^{-1}$	M1	Applies $\mathbf{A}=\mathbf{PDP}^{-1}$
	$\mathbf{A}=\frac{1}{b+1}\begin{pmatrix}1&-1&0\\1&0&1\\0&b&-1\end{pmatrix}\begin{pmatrix}2b&2&2\\-1&1&1\\-3b&3b&-3\end{pmatrix}$	M1 A1	Multiplies two adjacent matrices. $\mathbf{PD}=\begin{pmatrix}2&-1&0\\2&0&3\\0&b&-3\end{pmatrix}$
	$\frac{1}{b+1}\begin{pmatrix}1+2b&1&1\\-b&3b+2&-1\\2b&-2b&3+b\end{pmatrix}$	A1	

Question	Answer	Marks	Guidance
11E(i)	**Alternative method for 11E(i)**		
	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$	**B3**	Matches eigenvalues with eigenvectors to form systems of equations. B1 for each correctly matched pair.
	Three values found correctly from solving systems of equations. Another three values found correctly from solving systems of equations. Attempting to find final three values from systems of equations.	**M1 A1** **M1 A1** **M1**	
	$\mathbf{A} = \frac{1}{b+1}\begin{pmatrix} 1+2b & 1 & 1 \\ -b & 3b+2 & -1 \\ 2b & -2b & 3+b \end{pmatrix}$	**A1**	Final answer given in matrix form.
		9	
11E(ii)	$\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	**M1 A1**	M1 for $\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
		2	

Question	Answer	Marks	Guidance
11E(iii)	$\mathbf{A}^n\begin{pmatrix}1\\1\\0\end{pmatrix}=2^2\begin{pmatrix}1\\1\\0\end{pmatrix}\Rightarrow n=2$	B1	Uses $\mathbf{A}^n\begin{pmatrix}1\\1\\0\end{pmatrix}=2^n\begin{pmatrix}1\\1\\0\end{pmatrix}$
	$\mathbf{A}^n\begin{pmatrix}-1\\0\\b\end{pmatrix}=\begin{pmatrix}-1\\0\\b^{-1}\end{pmatrix}\Rightarrow b=b^{-1}\Rightarrow b=1$	M1 A1	M1 for stating $b=b^{-1}$ (since $\mathbf{A}^n\begin{pmatrix}-1\\0\\b\end{pmatrix}=\begin{pmatrix}-1\\0\\b\end{pmatrix}$). Must be only one answer ($b=1$) for A1.
		3	

Question	Answer	Marks	Guidance
11O(i)(a)	$\ln y = t \ln a$ $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = \ln a$	M1 A1	Differentiates both sides.
	$\frac{\mathrm{d}y}{\mathrm{d}t} = y \ln a = a^t \ln a$	A1	AG
		3	
11O(i)(b)	$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = a^t (\ln a)^2$	B1	
11O(ii)	$r = \ln a$	B1	States common ratio.
	$-1 < \ln a < 1 \Rightarrow \mathrm{e}^{-1} < a < \mathrm{e}$	M1 A1	Uses convergence condition for a geometric series.
		3	

Question	Answer	Marks	Guidance
11O(iii)	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{a^t \ln a}{at^{a-1}}=\left(\frac{\ln a}{a}\right)\left(\frac{a^t}{t^{a-1}}\right)$	M1 A1	Uses chain rule to find $\frac{dy}{dx}$
	$\frac{d}{dt}\left(\frac{dy}{dx}\right)=\left(\frac{\ln a}{a}\right)\left(\frac{t^{a-1}a^t \ln a-(a-1)t^{a-2}a^t}{t^{2(a-1)}}\right)$	M1 A1	Uses quotient or product rule to find $\frac{d}{dt}\left(\frac{dy}{dx}\right)$
	$\frac{d^2y}{dx^2}=\left(\frac{\ln a}{a^2}\right)\left(\frac{t^{a-1}a^t \ln a-(a-1)t^{a-2}a^t}{t^{3(a-1)}}\right)$ $=\left(\frac{\ln a}{a^2}\right)\left(\frac{\ln a-(a-1)t^{-1}}{t^{2(a-1)}}\right)a^t$	M1 A1	Uses chain rule to find $\frac{d^2y}{dx^2}$
	$t=2\Rightarrow\frac{d^2y}{dx^2}=2^{1-2a}\ln a(2\ln a-a+1)$	A1	Substitutes $t=2$, AG.
		7	