FURTHER MATHEMATICS **9231/21**

Paper 2 **October/November 2017**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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This document consists of **15** printed pages.

Mark Scheme Notes

Marks are of the following three types:

M   Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A   Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B   Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol FT implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

  - Note: B2 or A2 means that the candidate can earn 2 or 0.
    B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF/OE   Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG   Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO   Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO   Correct Working Only – often written by a 'fortuitous' answer

ISW   Ignore Subsequent Working

SOI   Seen or implied

SR   Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1   A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR –2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1   This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question	Answer	Marks	Guidance
1	$-p + 2t = 0$, $p = 4$	**M1 A1**	Find p by equating transverse acceln. to 0 at $t = 2$
	$a_R = (8 - pt + t^2)^2 / 0{\cdot}8$	**M1**	Find radial acceleration a_R at $t = 2$ in terms of p from v^2/r
	$= 4^2 / 0{\cdot}8 = 20$ [m s^{-2}]	**A1**	Evaluate with $p = 4$
		4	

Question	Answer	Marks	Guidance
2	$(3\pi/5)^2 = \omega^2 (a^2 - (a - 1{\cdot}6)^2)$ (AEF)	**M1 A1**	Use $v^2 = \omega^2 (a^2 - x^2)$ in each posn. (**M1** for either)
	$(\pi/4)^2 = \omega^2 (a^2 - (a - 0{\cdot}2)^2)$ $(3\pi/5)^2 (0{\cdot}4a - 0{\cdot}04) = (\pi/4)^2 (3{\cdot}2a - 2{\cdot}56)$	**A1**	Combine to find amplitude a and ω^2 (or ω)
	$a = 2{\cdot}6$ [m]	**M1 A1**	(**M1** for either)
	$\omega^2 = (\pi/4)^2$ *or* $\omega = \pi/4$ (AEF)	**A1**	Find other unknown
	$T = 2\pi / (\pi/4) = 8$ [s]	**B1FT**	Find period T from $T = 2\pi/\omega$ ($\surd$ on ω^2 or ω)
		7	

Question	Answer	Marks	Guidance
3(i)	$mv_A + kmv_B = 2mu + kmu$ (AEF) $[v_A + kv_B = 2u + ku]$	M1	Use conservation of momentum for A & B (allow omission of m in all momentum eqns)
	$v_B - v_A = ½ (2u - u)$ $[= ½ u]$	M1	Use Newton's restitution law with consistent LHS signs
	$v_B = u (2k + 5) / 2(k + 1)$ *or* $u (k + 5/2) / (k + 1)$ (AEF)	A1	Combine to find v_B
	$[v_A = u (k + 4) / 2(k + 1)]$ $v_B > 4u/3$ if $k < 7/2$	M1 A1	Find inequality for k from speeds of B and C after 1st collision
		5	
3(ii)	$kmw_B + mv_C = kmv_B + m(4u/3)$ (AEF) $[2w_B + v_C = 2(3u/2) + 4u/3$ $= 13u/3$ when $k = 2]$	M1	Use conservation of momentum for B & C
	$v_C - w_B = ½ (v_B - 4u/3)$ $[= u/12]$ $(k + 1) w_B = (k - ½) v_B + 2u$	M1	Use Newton's restitution law with consistent LHS signs Combine to find w_B
	$3 w_B = (3/2) v_B + 2u$ with $v_B = 3u/2$, so $w_B = 17u/12$	*A1	when $k = 2$
	$v_A = u$, $v_A < w_B$	DB1	Verify no further collisions between A and B

	EITHER: $(k + 1) v_C = (3k/2) v_B + (2 - k) (2u/3)$
	         $3 v_C = 3 v_B$ with $v_B = 3u/2$ so $v_C = 3u/2 > w_B$
		(DB1)	*EITHER:* Find v_C and verify no further collisions
			         between B and C

	OR: B and C cannot meet again since they move apart after
	        colliding (AEF)
		(DB1)	*OR:* State explicitly that no further collisions
			          between B and C

		5	

Question	Answer	Marks	Guidance
4(i)	$T \times 2a = (5W/2) \times a \cos \theta$	**M1**	Take moments for rod about A
	$T = \frac{1}{2} (5W/2) \times (4/5) = W$	**A1**	to find tension T
		2	
4(ii)	$T \sin \theta = \mu (W + T \cos \theta)$	**M1**	Use $F_P = \mu R_P$ at P
	$(3/5) W = \mu (1 + 4/5)W = \mu (9/5)W,\ \mu = 1/3$	**A1**	to find μ
		2	

4(iii)	*EITHER:* $[\pm] X = T \sin \theta = 3W/5\ or\ 0.6W$
	           $[\pm] Y = 5W/2 - T \cos \theta\ or\ 5W/2 + W - R_P$
		(B1	*EITHER:* Find horizontal component X of force at A
			             Find vertical component Y of force at A

	$= 17W/10\ or\ 1.7W$	**B1)**	
	OR: $[\pm] X = (5W/2) \sin \theta = 3W/2\ or\ 1.5W$	**(B1**	*OR:* Find component X of force at A along BA
	$[\pm] Y = (5W/2) \cos \theta - T = W$	**B1)**	Find component Y of force at A perp. to BA
	$R_A{}^2 = X^2 + Y^2 = 13W^2/4,\ R_A = \frac{1}{2}W\sqrt{13}\ or\ 1.80\ W$	**B1FT**	Find magnitude of resultant force R_A at A (FT on X, Y)
		3	
4(iv)	$PB = (a + 2a \sin \theta) / \cos \theta = 5a/4 + 3a/2 = 11a/4\ or\ 2.75a$ $or\ x = 3a/4\ or\ 0.75a$	**B1**	Find length PB or extension x of string
	$T = \lambda (PB - 2a)/2a,\ \lambda = 8W/3\ or\ 2.67W$	**M1 A1**	Find modulus λ using Hooke's Law
		3	

Question	Answer	Marks	Guidance
5(i)	$A_{ABCD} = 24a^2$, $A_{EFGH} = 8a^2$, $A_{Frame} = 16a^2$ $M_{ABCD} = 3m/2$ and $m_{EFGH} = m/2$	**B1**	Use areas to find masses M_{ABCD} and m_{EFGH}
	$I_{ABCD} = \frac{1}{3} M_{ABCD}((3a)^2 + (2a)^2)$ $\quad\quad [= (13/3) M_{ABCD}\, a^2 = (13/2)\, ma^2]$	**B1**	Find MI of $ABCD$ about axis at centre Q
	$I_{EFGH} = \frac{1}{3} m_{EFGH}((2a)^2 + a^2)$ $\quad\quad [= (5/3)\, m_{EFGH}\, a^2 = (5/6)\, ma^2]$	**B1**	Find MI of $EFGH$ about axis at centre Q
	$I_{ABCD} - I_{EFGH} + m \times (6a)^2 \ [= (121/2 - 113/6)\, ma^2$ $\quad\quad$ or $(17/3 + 36)\, ma^2 = (125/3)\, ma^2]$	**M1 A1**	Find MI of frame about axis at O Result also follows from combining four rectangular parts
	$I_{Object} = (11m/12) \times (4a)^2 \quad\quad [= (44/3)\, ma^2]$	**B1**	Find MI of small object about axis at O
	$I = (13/2 - 5/6 + 36 + 44/3)\, ma^2 = (169/3)\, ma^2 \quad\quad$ AG	**A1**	Combine to verify MI of system about axis at O
		7	
5(ii)	$[-]\, I\, d^2\theta/dt^2$ or $[-]\, I\alpha$ $\quad\quad = mg \times 6a \sin\theta + (11/12)mg \times 4a \sin\theta$ $\quad$ or $(23/12)mg \times (116/23)a \sin\theta$ $\quad [= (29/3)\, mga \sin\theta]$	**M1 A1**	Use eqn of circular motion to relate $d^2\theta/dt^2$ to $\sin\theta$, where θ is angle of QO with vertical
	$d^2\theta/dt^2$ or $\alpha = -(29g / 169a)\, \theta$ $\quad$ or $-(0\cdot172\, g / a)\, \theta$	**M1 A1**	Approximate $\sin\theta$ by θ to show SHM (**M0** if wrong sign or $\cos\theta \approx \theta$ used)
	$T = 2\pi / \sqrt{(29g/169a)}$ $\quad = 26\pi \sqrt{(a/29g)}$ or $15\cdot2\sqrt{(a/g)}$ or $4\cdot80\sqrt{a}$ (AEF)	**A1**	Find period T from $T = 2\pi/\omega$ (**A1** requires some simplification)
		5	

Question	Answer	Marks	Guidance
6(i)	$p = (1/6)^2$ *or* $1/36$	**B1**	Find (or imply) probability p of pair of sixes in one throw
	$1/p = 36$	**B1**	Find mean value of X
6(ii)	$P(X = 12) = p\,(1 - p)^{11} = 0{\cdot}0204$	**M1 A1**	Find prob. of needing exactly 12 throws
6(iii)	$P(X > 12) = (1 - p)^{12} = 0{\cdot}713$	**M1 A1**	Find prob. of needing more than 12 throws
		2	

Question	Answer	Marks	Guidance
7(i)	$F(x) = \int f(x)\,dx = -e^{-0.2x} + c = 1 - e^{-0.2x}\ (x \geq 0)$	**M1**	State, or integrate and use $F(0) = 0$ or $F(x) \to 1$ as $x \to \infty$
	and $F(x) = 0$ ($x < 0$ *or* otherwise)	**A1**	to find, $F(x)$ (**A0** if case $x < 0$ omitted)
		2	
7(ii)	$P(X > 2) = 1 - F(2) = e^{-0.4} = 0{\cdot}670$	**M1 A1**	Find $P(X > 2)$: (**M0** for $F(2)$)
		3	
7(iii)	$1 - e^{-0.2m} = \tfrac{1}{2}$, $e^{0.2m} = 2$	**M1**	Find median value m from $F(m)$ *or* $1 - F(m) = \tfrac{1}{2}$
	$m = 5 \ln 2$ *or* $3{\cdot}47$	**M1 A1**	
		3	

Question	Answer	Marks	Guidance
8	H_0: No association *or* method is independent of gender (AEF)	**B1**	State (at least) null hypothesis
	E_i: 9·0 19·8 31·2 6·0 13·2 20·8	**M1 A1**	Find expected values E_i (**A0** if rounded to integers)
	$\chi^2 = 0·111 + 3·073 + 1·482$ $+ 0·167 + 4·610 + 2·223$ (to 2 d.p.) $= 11·7$ (*or* 12·3)	**M1 A1**	Find value of χ^2 from $\Sigma\,(E_i - O_i)^2 / E_i$ [or $\Sigma\,O_i^2/E_i - n$] (allow 12·3 if integer values of E_i used)
	$\chi_{2,\,0.99}^{\,2} = 9·21$	**B1**	State or use correct tabular χ^2 value
	Reject H_0 if $\chi^2 >$ tabular value (AEF)	**M1**	Valid method for reaching conclusion
	11·7 $[\pm\,0·1] > 9·21$ so there is an association	**A1**	Correct conclusion, from correct values
		8	

Question	Answer	Marks	Guidance
9(i)	$r = S_{xy} / \sqrt{(S_{xx}\,S_{yy})}$ with e.g. $S_{xy} = 212·62 - 35·9 \times 36·8/8 = 47·48$ (*or* 5·935) $S_{xx} = 216·47 - 35·9^2/8 = 55·37$ (*or* 6·921) $S_{yy} = 244·96 - 36·8^2/8 = 75·68$ (*or* 9·46) (all to 3 s.f.)	**M1 A1**	Find correlation coefficient r (Insufficient working loses first **A1**)
	$r = 0·733$	***A1**	
		3	

Question	Answer	Marks	Guidance
9(ii)	H_0: $\rho = 0$, H_1: $\rho > 0$	**B1**	State both hypotheses (**B0** for r …)
	EITHER: $r_{8,\,1\%} = 0.789$	**(*B1**	State or use correct tabular one-tail r-value
	Accept H_0 if $|r| <$ tab. r-value (AEF)	**M1)**	State or imply valid method for conclusion
	OR: $t_r = r\sqrt{((n\text{-}2)\,/\,(1 - r^2))} = 2.64$, $t_{6,0.99} = 3.143$	**(*B1**	
	Accept H_0 if $|\,t_r\,| <$ tab. t-value (AEF)	**M1)**	
	No positive correlation (AEF)	**DA1**	
		4	
9(iii)	$r_{14,\,1\%} = 0.661$, $r_{15,\,1\%} = 0.641$ so $n_{min} = 15$	**M1 A1**	Find n_{min} from relevant two-tail tabular value[s] SC**:** Award **B1** for stating 15 without any justification
		2	

Question	Answer	Marks	Guidance
10(i)	$\bar{x} = 0.91$, $\bar{y} = 1.205$ $s_X^2 = 19.56 / 49$ [$= 489/1225$ or 0.3992] *and* $s_Y^2 = 30.25 / 59$ [$= 121/236$ or 0.5127]	B1 M1	Find both sample means Estimate both population variances (allow biased here: 0.3912 *and* 0.5042)
	EITHER: $s^2 = s_X^2/50 + s_Y^2/60$	(M1	Estimate or imply combined variance
	$= 0.01653$ *or* 0.1286^2 (to 3 s.f. throughout)	A1	
	[$\pm$] ($\bar{y} - \bar{x}$) $\pm z\, s$	M1	Find confidence interval for difference $Y - X$ *or* $X - Y$
	$z_{0.95} = 1.645$ [$\pm$] 0.295 ± 0.211 (allow 0.212)	A1	Use appropriate tabular value Evaluate confidence interval (either form)
	or [$\pm$] [0.084, 0.506] (allow [$\pm$] [0.083, 0.507])	A1)	

	OR: Assume equal [population] variances
	      $s^2 = (49\, s_X^2 + 59\, s_Y^2) / 108$
	       or $(19.56 + 30.25) / 108$
		(B1	State assumption
			 Find or imply pooled estimate of common variance
			 (note s_X^2 and s_Y^2 not needed explicitly so first M1 may be implied by result)

	$= 4981/10800$ *or* 0.461 *or* 0.679^2	B1	
	[$\pm$] ($\bar{y} - \bar{x}$) $\pm z\, s\, \sqrt{(1/50 + 1/60)}$	M1	Find confidence interval for difference $Y - X$ *or* $X - Y$
	$z_{0.95} = 1.645$	A1	Use appropriate tabular z-value (or t-value from calculator)
	[$\pm$] 0.295 ± 0.214 *or* [$\pm$] [0.081, 0.509]	A1)	Evaluate confidence interval (either form)
		7	

Question	Answer	Marks	Guidance
10(ii)	$z = (1 \cdot 205 - 0 \cdot 91) /s = 0 \cdot 295 /s$ $\quad = 2 \cdot 29[4] \quad$ [or $2 \cdot 26[9]$]] (to 3 s.f.)	**M1 A1**	Find value of z (either sign)
	$(z) = 0 \cdot 989[1]$ [or $0 \cdot 988[4]$]	**A1**	Find $\Phi(z)$
	$100 \times (1 - 0 \cdot 989) \times 2 = 2 \cdot 2$ [or $2 \cdot 3$] (to 1 d.p.)	**M1 A1**	Find limiting value for α, based on two-tail test (**M0** for basing on one-tail test)
	$\alpha < (or \leqslant) \ 2 \cdot 2$ [or $2 \cdot 3$]	**A1**	Find set of possible values of α (Treat α instead of α% as misread)
		6	

Question	Answer	Marks	Guidance
11A(i)	$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 - mga \, (\cos \alpha + \cos \beta)$	**M1 A1**	Find v^2 at A' from conservation of energy (**A0** if no m)
	$mv^2/a = mg \cos \beta$ $u^2 = ag \cos \beta + 2ag \, (1/16 + \cos \beta)$	**B1**	Use $F = ma$ radially at A' with $R_{A'} = 0$ Use $\cos \alpha = 1/16$ and eliminate v^2 to verify u^2
	$\quad = (1/8) \, ag \, (1 + 24 \cos \beta)$ AG	**M1 A1**	
		5	

Question	Answer	Marks	Guidance
11A(ii)	$\frac{1}{2}mw^2 = \frac{1}{2}mu^2 - mga\,(1 + \cos 2\beta)$	**B1**	Find w^2 at B' from conservation of energy (**B0** if no m)
	$mw^2/a = mg \cos 2\beta$ $u^2 = ag \cos 2\beta + 2ag\,(1 + \cos 2\beta)$	**B1**	Use $F = ma$ radially at B' with $R_{B'} = 0$ Eliminate w^2 to find u^2
	$\quad = ag\,(2 + 3 \cos 2\beta)$	**M1**	
	$1 + 24 \cos \beta = 8\,(2 + 3 \cos 2\beta)$ $\qquad\quad = 16 + 48 \cos^2 \beta - 24$ $16 \cos^2 \beta - 8 \cos \beta - 3 = 0$	**M1**	Combine eqns for u^2 Formulate and solve quadratic to find $\cos \beta$
	$\cos \beta = \frac{3}{4}$ [rejecting $-\frac{1}{4}$]	**M1 A1**	
		6	
11A(iii)	$u^2 = (19/8)\,ag$ or $2\cdot375\,ag$ $R = mu^2/a + mg \cos \alpha$ $\quad = (19/8 + 1/16)\,mg$	**B1**	Find u^2 using value of $\cos \beta$ Use $F = ma$ radially at A to find reaction R at A
	$\quad = (39/16)\,mg$ or $2\cdot44\,mg$	**M1 A1**	
		3	

Question	Answer	Marks	Guidance
11B(i)	e.g. $S_{xy} = 218{\cdot}72 - 45{\cdot}3 \times 40{\cdot}5/9 = 14{\cdot}87$ *or* $1{\cdot}652$ $S_{xx} = 245{\cdot}59 - 45{\cdot}3^2/9 \qquad = 17{\cdot}58$ *or* $1{\cdot}953$ *or* $S_{yy} = 195{\cdot}11 - 40{\cdot}5^2/9 \qquad = 12{\cdot}86$ *or* $1{\cdot}429$		Find reqd. values
	EITHER: $b_1 = S_{xy} / S_{xx} = 14{\cdot}87 / 17{\cdot}58 = 0{\cdot}84585 = 0{\cdot}846$ $(y - 40{\cdot}5/9) = b_1 (x - 45{\cdot}3/9)$ $(y - 4{\cdot}5) = 0{\cdot}846 (x - 5{\cdot}033)$	**(M1 A1**	Find gradient in $y - \bar{y} = b_1 (x - \bar{x})$ to 3 s.f. Find eqn. of regression line to 3 s.f.
	$y = 0{\cdot}846x + 0{\cdot}24257 = 0{\cdot}846x + 0{\cdot}243$	**M1 A1**	
	$x = 4{\cdot}68$	**A1)**	Find x when $y = 4{\cdot}2$
	OR: $b_2 = S_{xy} / S_{yy} = 14{\cdot}87 / 12{\cdot}86 = 1{\cdot}1563 = 1{\cdot}16$ $(x - 45{\cdot}3/9) = b_2 (y - 40{\cdot}5/9)$ $(x - 5{\cdot}033) = 1{\cdot}16 (y - 4{\cdot}5)$	**(M1 A1**	Find gradient in $x - \bar{x} = b_2 (y - \bar{y})$ to 3 s.f. Find eqn. of regression line to 3 s.f.
	$x = 1{\cdot}16y - 0{\cdot}170[04]$	**M1A1**	
	$x = 4{\cdot}69$	**A1)**	Find x when $y = 4{\cdot}2$ (**A0** for $x = 4{\cdot}70$)
		5	

Question	Answer	Marks	Guidance
11B(ii)	H_0: $\mu_x - \mu_y = 0.3$, H_1: $\mu_x - \mu_y > 0.3$ (AEF)	**B1**	State hypotheses (**B0** for $\bar{x}$ …)
	$\bar{d} = 4.8 / 9$ *or* $8/15$ *or* 0.533 (where $d = x - y$)	**B1**	Find sample mean
	$s^2 = (3.26 - 4.8^2/9) / 8$ $= 7/80$ *or* 0.0875 *or* 0.296^2	**M1 A1**	Estimate population variance (allow biased here: $7/90$ *or* 0.0778 *or* 0.279^2)
	$t = (\bar{d} - 0.3) / (s/\sqrt{9}) = 2.37$	**M1 A1**	Find value of t
	$t_{8, 0.975} = 2.306$ *or* 2.31	**B1**	State or use correct tabular t-value (or can compare $\bar{d}$ with $0.3 + t_{8, 0.975}$ s$/\sqrt{9} = 0.527$)
	Reject H_0 if $t >$ tabular value (AEF) 2.37 $[\pm 0.1] > 2.31$ so accept belief	**M1**	Valid method for reaching conclusion Correct conclusion, from correct values
	[of increase of more than 0.3] (AEF)	**A1**	SC: Wrong (hypothesis) test can earn only **B1** for hypotheses
		9	