FURTHER MATHEMATICS

Paper 9231/11
Paper 11

Key messages

- Candidates should write down every step of working, particularly when proving a given result. They should make sure that their solutions cover all possibilities. Induction proofs particularly should include all the necessary steps and statements
- Candidates should make sure that sketch graphs show all the necessary key points, including intercepts on axes.
- Candidates need to learn any formulae from both 9231 and 9709 so that they can choose and apply the correct formula to a problem.
- Candidates should check their answers to correct slips such as using the wrong variable or missing terms in implicit differentiation

General comments

Candidates completed the paper, and most scripts showed excellent algebraic handling and calculus skills. Answers were generally well presented and usually with the appropriate amount of detail. Most candidates took care over their calculations avoiding loss of marks through careless errors. The majority of candidates showed very good knowledge across the whole syllabus.

Comments on specific questions

Question 1

Most candidates multiplied out correctly and recognised which formulae they needed to complete the summation, although there were some who did not get the final term $\sum_{r=1}^{n} 3$ correct. Algebraic manipulation to reduce the expression to three terms was well done.

Answer: $\frac{n}{3}\left(16n^2 + 12n - 13\right)$

Question 2

Most candidates remembered the correct form of the complementary function and particular integral, although did not include all three terms to form the quadratic for their particular integral. Variables usually matched those used in the question.

Answer: $x = e^{-t}\left(A\cos 2t + B\sin 2t\right) + \frac{22}{25} + \frac{4}{5}t - t^2$

Question 3

(i) Stronger candidates were able to present a structured proof of the given result, whilst others recognised the first derivative of $x^{n+1}\ln x$ on the right hand side of the equation, but did not manipulate the derivatives correctly.

(ii) This induction proof proved challenging. Candidates started well, but failed to separate the two parts of $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^k + [k+1]x^k \ln x\right)$ so that they could apply the formula. Once they had separated the parts, good candidates realised that $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^k\right) = k!$. The strongest candidates gave a full concluding statement making clear both the base case and that if the statement were true for *k*, it was also true for *k*+1. Both statements are necessary to complete the proof.

Question 4

(i) This question was very well done by most candidates. Some chose to multiply out and group the terms before using the properties of the roots of the original equation, others used a substitution to form a related cubic.

(ii) A range of correct solutions was seen, with some candidates multiplying out the brackets and grouping terms, others using the fact that $(\beta + y) = \left(1\frac{1}{2} - \alpha\right)$ and so on, and others using $x = \left(1\frac{1}{2} - y\right)$ as a substitution and forming a new cubic equation.

Answers: **(i)** $9\frac{1}{2}$ **(ii)** –2

Question 5

(i) This question was very well done by most candidates, who differentiated the expression implicitly before rearranging and using the given conditions to find *x* and *y*. Errors were occasionally made with the final term

Answers: **(i)** (2, –2) **(ii)** $y'' = \frac{1}{2}$

Question 6

(i) Many candidates knew and used the formula $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{BC}|$ to find the area of the triangle, though a number divided the vector representing one side by 3 rather than using the vector itself. Others used the scalar product to find one angle of the triangle before going on to find the area. This method did give rise to some arithmetic slips. One candidate used Hero's formula – a neat and direct method.

(ii) A range of methods was used here, with many candidates remembering and using the cross-product formula correctly. Others found the point on BC at the foot of the perpendicular and hence the length of the perpendicular. Unfortunately some candidates tried to use the scalar triple product which was not applicable to this problem.

(iii) Most candidates were able to find the equation of the plane, and gave their answer in cartesian form as required.

Answers: **(i)** 13.9 or $\frac{3\sqrt{86}}{2}$ **(ii)** 4.15 or $\frac{\sqrt{430}}{5}$ **(iii)** $7x + y + 6z = 19$

Question 7

(i) The majority of candidates were able to reduce the matrix and find the rank. Although there were some sign errors and some arithmetical slips, most were then able to extract and manipulate the equations to find two vectors that formed a basis for the null space.

(ii) Most candidates multiplied the matrix and given vector correctly. Some recognised that this vector formed the particular solution and combined it with a general element from the null space correctly. The best scripts fully justified this.

Answers: **(i)** For example $\left\{\lambda\begin{pmatrix}-1\\-3\\1\\0\end{pmatrix}, \mu\begin{pmatrix}-8\\-5\\0\\1\end{pmatrix}\right\}$ or $\left\{\lambda\begin{pmatrix}19\\0\\5\\3\end{pmatrix}, , \mu\begin{pmatrix}0\\19\\-8\\1\end{pmatrix}\right\}$

(ii) $A\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}=\begin{pmatrix}3\\21\\24\\27\end{pmatrix}$ and $x=\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\-3\\1\\0\end{pmatrix}+\mu\begin{pmatrix}-8\\-5\\0\\1\end{pmatrix}$ or equivalent

Question 8

(i) Most candidates remembered how to integrate this function and almost all showed the function as necessary before substituting limits.

(ii) The majority of candidates split the integrand and used integration by parts correctly. Some did not justify the value $2^{\frac{1}{2}n-1}$ but most made each step very clear on the way to the given answer. It proved easier to work with $\sec\theta$ and $\tan\theta$ than revert to $\cos\theta$ and $\sin\theta$, but other methods were used successfully too.

(iii) Those who remembered the formula correctly did very well on this part, others were still able to pick up marks for using the given reduction formula. Others embarked on complicated integrations that did not use the result of **part (ii)** as they should have expected.

Answers: **(i)** 1 **(iii)** $\frac{28\pi}{15}$

Question 9

(i) Although most candidates found the equations of the two vertical asymptotes, a significant number failed to realise that $y = 0$ formed a horizontal asymptote. Some thought $y = 3$ formed an asymptote, and others thought that $y = 3x - 9$ formed an oblique asymptote.

(ii) Candidates who formed a quadratic in x and then found its discriminant scored well, though some did not fully justify the given solution. Many of those who tried to use the turning points did not give a complete answer as they did not consider **why** these values were limits for the range.

(iii) Most were able to differentiate either the function as given in the question, or by expressing it in partial fractions. Some used implicit differentiation successfully too. There were some errors seen by those applying the quotient rule which needs to be part of every candidate's problem solving skill set. Others tried to use the restriction on the range found in **(ii)** without fully justifying why these were turning points.

(iv) Some candidates took care to include all information. Most found the intercepts with the axes algebraically, but some added further intercepts when sketching. A common error was to omit the left hand branch of the graph. Some graphs did not represent functions.

Answers: **(i)** $x = -1$, $x = 2$ and $y = 0$ **(iii)** (1,3) and $\left(5, \frac{1}{3}\right)$ **(iv)** Asymptotes, maximum and minimum points, intercepts at (0, 4.5) and (3,0).

Question 10

(i) Most candidates used de Moivre's theorem directly to find the given result, a few used $z - \frac{1}{z} 2\text{isin}(\theta)$ which took considerable manipulation. Although it was possible to obtain the result through using trigonometric relationships, this did not meet the criterion of the question.

(ii) Following the instruction 'hence', the strongest candidates explained why the roots were the given pairs, either by using the related quadratic equation or by ensuring that the four roots they gave were distinct. They also justified the omission of $\sin(\theta) = 0$ as a solution. Some weaker solutions included errors such as $n\left(\frac{4\pi}{5}\right) = -\sin\left(\frac{\pi}{5}\right)$.

(iii) Candidates using the related quadratic equation produced some good solutions to this part, others were able to relate $n\left(\frac{4\pi}{5}\right)$ to $-\sin-\left(\frac{\pi}{5}\right)$ correctly, but mistakes made above were repeated here leading some candidates to reach the correct answers from wrong working.

Answers: **(iii)** $\frac{5}{16}$ and $\frac{5}{4}$

Question 11

EITHER

(i) The majority of candidates who tackled this option knew the relationship between a matrix and its eigenvalues and eigenvectors and were able to use the rules of matrix multiplication to prove the given result. There were some who added extra eigenvectors or changed the order of multiplication.

(ii) Candidates were able to find the eigenvalues using the characteristic equation, and went to find the eigenvectors by their preferred method. This part of the question was also very well done, with few numerical errors seen

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Answers: **(ii)** –1, 2 and 3 with corresponding eigenvectors $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$

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OR

This was a slightly less popular choice.

(i) Sketches were generally clear and most candidates labelled the intercept on the initial line correctly, though some omitted the constant, a. Some cardioids were rotated or translated, but most were correctly positioned.

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Answers: **(iii)** $\frac{a^2}{16}\left(4\pi + 9\sqrt{3}\right)$ **(iv)** $2a$

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

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Paper 9231/13
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Answers: **(iii)** $\frac{a^2}{16}\left(4\pi + 9\sqrt{3}\right)$ **(iv)** $2a$

FURTHER MATHEMATICS

Paper 9231/21
Paper 21

Key messages

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Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many, but not all. When relevant in Mechanics questions it is helpful to show on a diagram what forces are acting and also their directions as in **Question 4** and the directions of motion of particles, as in **Question 3**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 9 and 11** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable in such cases to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances, for example.

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Most candidates first found the value 4 for p correctly by equating the transverse acceleration $2t - p$ to zero with $t = 2$. This value should then be used in the standard formula v^2/r to find the required magnitude of the radial component of the acceleration, though it was not unusual to see the formula written correctly but v used in place of v^2 in the ensuing calculation. Candidates should be able to distinguish between linear and angular measures of the radial acceleration, with consideration of only the former needed here. A symptom of such confusion was the occasional statement of the units of the linear radial acceleration in terms of radians rather than metres.

Answer: 20 m s^{-2}

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The obvious approach here is to effectively apply the standard SHM formula $v^2 = \omega^2 (a^2 - x^2)$ at each of the two points specified in the question. The corresponding values of the displacement x from the centre of motion are not, however, 1·6 and 0·2, but rather $a - 1{\cdot}6$ and $a - 0{\cdot}2$. Solution of the resulting two simultaneous equations yields both the required amplitude a and also the value $\frac{1}{4}\pi$ of ω. As most candidates were aware, the period may then be found from $2\pi/\omega$.

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Answer: **(i)** $k < 7/2$

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Answers: **(i)** W; **(ii)** $\frac{1}{3}$; **(iii)** $\frac{1}{2}W\sqrt{13}$; **(iv)** $8W/3$.

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ABCD and the removed lamina *EFGH*, but it is equally permissible to regard the frame as composed of two pairs of identical rectangular laminas forming the top, bottom and sides. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms for I with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. In the second part, the couple acting on the system should be found in terms of sin θ, where θ is the small angular displacement, and equated to $I\,d^2\theta/dt^2$. Approximating sin θ by θ yields the familiar form $d^2\theta/dt^2 = -\omega^2\theta$ of the standard SHM equation, in which the minus sign is essential, and from which the required period $2\pi/\omega$ is obtained. While this final result may be stated in a number of acceptable forms, $\sqrt{169}$ should certainly be simplified to 13. Candidates should be aware that the constant parameter ω in the SHM equation is not equal to $d\theta/dt$ even though the same symbol is sometimes used to denote the latter, and thus finding $d\theta/dt$ from conservation of energy at some arbitrary angular displacement and equating it to the SHM parameter ω is wholly invalid.

Answers: **(ii)** $26\pi\sqrt{(a/29g)}$ or $4{\cdot}80\sqrt{a}$.

Question 6

The mean value of X was widely known to equal $1/p$, where p is the probability of throwing a pair of sixes, namely 1/36, though p was occasionally wrongly thought to be 1/6. Most candidates also knew that the probabilities that exactly 12 throws, and more than 12 throws, are required to obtain a pair of sixes are respectively $p(1-p)^{11}$ and $(1-p)^{12}$, even when their value of p is incorrect.

Answers: **(i)** 36; **(ii)** 0·0204; **(iii)** 0·713.

Question 7

As well as integrating f(x) to find the distribution function F(x) over $x \geqslant 0$, candidates should preferably state the value of F(x) for other values of x. P($X > 2$) follows from 1 – F(2), and not of course F(2). Apart from the few candidates who confused median and mean, the method for finding the median value m of X by equating F(m) to ½ was well known.

Answers: **(i)** 0 ($x < 0$), $1 - e^{-0.2x}$ ($x \geqslant 0$); **(ii)** 0·670; **(iii)** 5 ln 2 or 3·47.

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Most candidates produced good answers to this question. After stating the hypotheses to be tested, a table of the expected method of voting against gender is produced in the usual way and preferably to an accuracy of one (or more) decimal places. The calculated χ^2-value 11·7 should be compared with the tabular value 9·21, leading to rejection of the null hypothesis. Thus the required answer to the question is that there is an association between method of voting and gender. Those candidates who are more familiar with the use of a χ^2–test for independence in a contingency table should be aware that independence corresponds to no association, rather than association.

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The required value of the correlation coefficient is readily found using the standard formula, and almost all candidates showed all the necessary working. Many also went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho > 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison of 0·733 with the tabular value 0·789 leads to a conclusion of there being no evidence of positive correlation. The final part requires a comparison of the given value 0·651 with tabulated two-tail 1% critical values of the product moment correlation coefficient, with some brief explanation to justify the answer.

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A key decision to be made is whether to use an estimate of a combined variance or a pooled estimate of a common population variance. In the absence of any information in the question about the variances of the two distributions, it is more appropriate to estimate and use a combined variance. However the alternative is

acceptable to the Examiners provided candidates state explicitly that they are assuming the two population variances to be equal. Most candidates knew how to find a confidence interval for the difference in the population means, though the final result differs slightly depending on which variance estimate is used. That given here is for the combined variance $s^2 = 0{\cdot}01653$ and a z-value of 1·645, used by the majority of candidates. The second part seemingly proved more challenging. It requires the calculation of an appropriate z-value such as $0{\cdot}295/s = 2{\cdot}294$ for which the table of the Normal distribution function gives $\Phi(z) = 0{\cdot}989$. When using this to find the set of possible values of α, candidates should be aware that testing for any difference in the means implies a two-tailed test.

Answers: **(i)** [0·084, 0·506]; **(ii)** $\alpha < 2{\cdot}2$.

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Answers: **(ii)** ¾; **(iii)** $39mg/16$.

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There is no strong reason here for preferring the regression line of y on x to that of x on y, and so different choices were made by different candidates. The equation of the required regression line follows from first finding the gradient using the standard formula and then utilising the sample means to find the constant term. The value of x when $y = 4{\cdot}2$ may be estimated by substituting this value of y in the regression line. Although the rubric specifies that non-exact numerical answers be given correct to 3 significant figures, candidates should retain additional figures in their intermediate working so that this level of accuracy is attained in the final answer. Most candidates understood the general approach to performing the test in the second part, even if it was not always carried out perfectly. As in all such tests, the hypotheses should be stated in terms of the population mean and not the sample mean. The unbiased estimate 7/80 = 0·0875 of the population variance may be used to calculate a t-value of 2·37. Since it is a one-tail test, comparison with the tabulated value of 2·306 leads to rejection of the null hypothesis, and thus the conclusion that the organiser’s belief is justified.

Answers: **(i)** $y = 0{\cdot}846x + 0{\cdot}243$, 4·68 or $x = 1{\cdot}16y - 0{\cdot}170$, 4·69.

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

Key messages

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FURTHER MATHEMATICS

Paper 9231/23
Paper 23

Key messages

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Answer: **(i)** $k < 7/2$

Question 4

Since much of the following working depends on the tension T, it is important that candidates take particular care over finding it correctly. T cannot be found in terms of W by a resolution of forces, and instead moments should be taken for the rod about A. While it was widely understood that the required coefficient of friction may be found from the ratio of the friction and normal reaction at P, incorrect values were often used for one or both of these forces instead of $T \sin \theta$ and $W + T \cos \theta$ respectively. When finding and combining two perpendicular components of the resultant force on the rod at the hinge, forces may be resolved either horizontally and vertically or along and normal to the rod. The final part requires the use of Hooke's Law, with the extension $\frac{3}{4}a$ of the string found by trigonometry.

Answers: **(i)** W; **(ii)** $\frac{1}{3}$; **(iii)** $\frac{1}{2}W\sqrt{13}$; **(iv)** $8W/3$.

Question 5

Verifying the given moment of inertia I presented most candidates with little difficulty, requiring the use of standard formulae and the parallel axes theorem to formulate and then sum the individual moments of inertia of the frame and particle. Most candidates treated the frame as the difference between the original lamina

ABCD and the removed lamina *EFGH*, but it is equally permissible to regard the frame as composed of two pairs of identical rectangular laminas forming the top, bottom and sides. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Candidates who simply write down a sum of terms for I with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. In the second part, the couple acting on the system should be found in terms of $\sin\theta$, where θ is the small angular displacement, and equated to $I\,d^2\theta/dt^2$. Approximating $\sin\theta$ by θ yields the familiar form $d^2\theta/dt^2 = -\omega^2\theta$ of the standard SHM equation, in which the minus sign is essential, and from which the required period $2\pi/\omega$ is obtained. While this final result may be stated in a number of acceptable forms, $\sqrt{169}$ should certainly be simplified to 13. Candidates should be aware that the constant parameter ω in the SHM equation is not equal to $d\theta/dt$ even though the same symbol is sometimes used to denote the latter, and thus finding $d\theta/dt$ from conservation of energy at some arbitrary angular displacement and equating it to the SHM parameter ω is wholly invalid.

Answers: **(ii)** $26\pi\sqrt{(a/29g)}$ or $4{\cdot}80\sqrt{a}$.

Question 6

The mean value of X was widely known to equal $1/p$, where p is the probability of throwing a pair of sixes, namely 1/36, though p was occasionally wrongly thought to be 1/6. Most candidates also knew that the probabilities that exactly 12 throws, and more than 12 throws, are required to obtain a pair of sixes are respectively $p(1-p)^{11}$ and $(1-p)^{12}$, even when their value of p is incorrect.

Answers: **(i)** 36; **(ii)** 0·0204; **(iii)** 0·713.

Question 7

As well as integrating $f(x)$ to find the distribution function $F(x)$ over $x \geqslant 0$, candidates should preferably state the value of $F(x)$ for other values of x. $P(X > 2)$ follows from $1 - F(2)$, and not of course $F(2)$. Apart from the few candidates who confused median and mean, the method for finding the median value m of X by equating $F(m)$ to ½ was well known.

Answers: **(i)** 0 $(x < 0)$, $1 - e^{-0.2x}$ $(x \geqslant 0)$; **(ii)** 0·670; **(iii)** $5\ln 2$ or 3·47.

Question 8

Most candidates produced good answers to this question. After stating the hypotheses to be tested, a table of the expected method of voting against gender is produced in the usual way and preferably to an accuracy of one (or more) decimal places. The calculated χ^2-value 11·7 should be compared with the tabular value 9·21, leading to rejection of the null hypothesis. Thus the required answer to the question is that there is an association between method of voting and gender. Those candidates who are more familiar with the use of a χ^2–test for independence in a contingency table should be aware that independence corresponds to no association, rather than association.

Question 9

The required value of the correlation coefficient is readily found using the standard formula, and almost all candidates showed all the necessary working. Many also went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho > 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison of 0·733 with the tabular value 0·789 leads to a conclusion of there being no evidence of positive correlation. The final part requires a comparison of the given value 0·651 with tabulated two-tail 1% critical values of the product moment correlation coefficient, with some brief explanation to justify the answer.

Answers: **(i)** 0·733; **(iii)** 15.

Question 10

A key decision to be made is whether to use an estimate of a combined variance or a pooled estimate of a common population variance. In the absence of any information in the question about the variances of the two distributions, it is more appropriate to estimate and use a combined variance. However the alternative is

acceptable to the Examiners provided candidates state explicitly that they are assuming the two population variances to be equal. Most candidates knew how to find a confidence interval for the difference in the population means, though the final result differs slightly depending on which variance estimate is used. That given here is for the combined variance $s^2 = 0{\cdot}01653$ and a z-value of 1·645, used by the majority of candidates. The second part seemingly proved more challenging. It requires the calculation of an appropriate z-value such as $0{\cdot}295/s = 2{\cdot}294$ for which the table of the Normal distribution function gives $\Phi(z) = 0{\cdot}989$. When using this to find the set of possible values of α, candidates should be aware that testing for any difference in the means implies a two-tailed test.

Answers: **(i)** [0·084, 0·506]; **(ii)** $\alpha < 2{\cdot}2$.

Question 11 (Mechanics)

This optional question was attempted by a minority of candidates, and many of those who did so produced good attempts. In the first part they correctly verified the given equation by conservation of energy between A and A' and also by equating the reaction at A' to zero The second part similarly requires both conservation of energy between B and B' and also that the reaction at B' be equated to zero. This enables the speed at B' to be eliminated, giving a second equation for u^2 which may be combined with the earlier one to yield a quadratic equation for $\cos\beta$, and hence its positive solution. Finally this enables the value of u^2 to be found, and used to determine the reaction R at A.

Answers: **(ii)** ¾; **(iii)** $39mg/16$.

Question 11 (Statistics)

There is no strong reason here for preferring the regression line of y on x to that of x on y, and so different choices were made by different candidates. The equation of the required regression line follows from first finding the gradient using the standard formula and then utilising the sample means to find the constant term. The value of x when $y = 4{\cdot}2$ may be estimated by substituting this value of y in the regression line. Although the rubric specifies that non-exact numerical answers be given correct to 3 significant figures, candidates should retain additional figures in their intermediate working so that this level of accuracy is attained in the final answer. Most candidates understood the general approach to performing the test in the second part, even if it was not always carried out perfectly. As in all such tests, the hypotheses should be stated in terms of the population mean and not the sample mean. The unbiased estimate 7/80 = 0·0875 of the population variance may be used to calculate a t-value of 2·37. Since it is a one-tail test, comparison with the tabulated value of 2·306 leads to rejection of the null hypothesis, and thus the conclusion that the organiser's belief is justified.

Answers: **(i)** $y = 0{\cdot}846x + 0{\cdot}243$, 4·68 or $x = 1{\cdot}16y - 0{\cdot}170$, 4·69.