Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/12**

Paper 1 **May/June 2017**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2017 series for most Cambridge IGCSE®, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

CAMBRIDGE
International Examinations

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol FT implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

  - Note: B2 or A2 means that the candidate can earn 2 or 0.
    B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)
CWO	Correct Working Only – often written by a 'fortuitous' answer
ISW	Ignore Subsequent Working
SOI	Seen or implied
SR	Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

<u>Penalties</u>

MR –1	A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR –2 penalty may be applied in particular cases if agreed at the coordination meeting.
PA –1	This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question	Answer	Marks	Guidance
1(i)	$\sum_{n+1}^{2n} u_r = (2n)^2(4n+3) - n^2(2n+3)$	**M1**	Method mark for using $S_{2n} - S_n$
	$= 14n^3 + 9n^2$	**A1**	
	Total:	**2**	
1(ii)	$u_r = r^2(2r+3) - (r-1)^2(2r+1)$	**M1A1**	Method mark for using $S_r - S_{r-1}$ OE
	$= 6r^2 - 1$	**A1**	SR: CAO **B1** without wrong working
	Total:	**3**	
2	Let P_n be the proposition that 5^n+3 is divisible by 4 $5^0 + 3 = 4 \Rightarrow P_0$ is true (allow P_1)	**B1**	Some explanation of what P_k being true means
	Assume that P_k is true for some non-negative integer k.	**B1**	or e.g. $5^k + 3 = 4\alpha$ for 2nd **B1**
	$5^{k+1} + 3 = 5(4\alpha - 3) + 3$	**M1**	*Alt method:* Use f(k+1) – f(k) **M1 A1**
	$= 20\alpha - 12 = 4(5\alpha - 3)$ (or shows that $5^{k+1} + 3 = 5.5^k + 5.3 - 4.3 = 5(5^k + 3) - 4.3$)	**A1**	
	P_0 is true and $P_k \Rightarrow P_{k+1}$, hence P_n is true for all non-negative integers n.	**A1**	
	Total:	**5**	

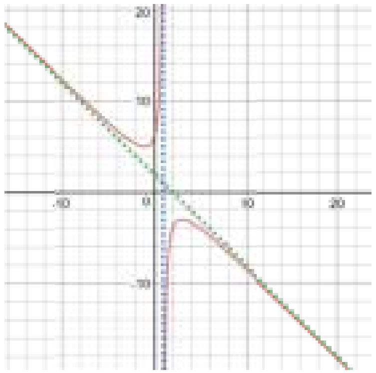
Question	Answer	Marks	Guidance
3(i)	$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1$ **or** $\sec^2 y\frac{dy}{dx}=1 \Rightarrow \left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1$ **or** $\frac{dy}{dx}=\cos^2 y$	**M1**	Using implicit differentiation
	$\Rightarrow 2x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1+x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2x\left(\frac{dy}{dx}\right)^2$ (AG)	**M1 A1**	M1 for good attempt at product rule
	Alt method: $\frac{\mathrm{d}y}{\mathrm{d}x}=\cos^2 y \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\cos y(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x}=-2x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$	**M1 A1**	M1 for good attempt at implicit differentiation
	Total:	**3**	
3(ii)	$y'(1)=\cos^2\left(\frac{\pi}{4}\right)\Rightarrow y'(1)=\frac{1}{2}$	**B1**	
	$\Rightarrow y''(1)=-\frac{1}{2}$	**B1 FT**	
	Total:	**2**	
4(i)	$\begin{vmatrix}1 & 3 & k\\ 2 & -1 & -5\\ 1 & 1 & 2\end{vmatrix}=0$	**M1 M1**	Using the determinant *Alt method:* Uses row operations
	$\Rightarrow k=8$	**A1**	
	Total:	**3**	

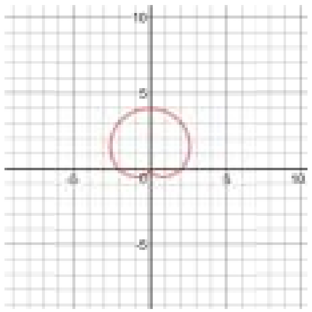
Question	Answer	Marks	Guidance
4(ii)	$x+3y+8z=4$ (1) $4x-2y-10z=-5$ (2)	**M1**	
	From (1) and (2) obtain $y=-3x$ or other correct expression $x+y+2z=1$ (3) Substitute $x=t$ and $y=-3t$ in (3) to obtain z.	**A1 FT**	**Alternative:** Find REF for augmented matrix and form equations (**M1**) Correctly (**A1**)
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{-2} \\ \frac{3}{2} \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ (OE)	**A1**	
	Total:	**3**	
5(i)	Eigenvectors are **i** + 2**j** + 2**k**, **i** + **k** and **j** + **k** (OE) for $\lambda = 1, -1$ and -2 respectively.	**M1A1**	
	(Award **M1A1** for any one correct and **A1** for each of the other two.)	**A1A1**	
	Total:	**4**	
5(ii)	Eigenvalues for **B** are –1, –3 and –4.	**B1**	
	Eigenvectors are the same as for **A** respectively.	**B1 FT**	
	Total:	**2**	

Question	Answer	Marks	Guidance
6(i)	$I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1}\cos x\,\mathrm{d}x$	M1A1	Uses integration by parts with $\mathrm{u} = x^n$
	$= 0 + \left[nx^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2}\sin x\,\mathrm{d}x$	A1	
	$= n\left(\frac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2} \Rightarrow I_n + n(n-1)I_{n-2} = n\left(\frac{1}{2}\pi\right)^{n-1}$ (AG)	A1	
	Total:	**4**	
6(ii)	$I_1 = \int_0^{\frac{\pi}{2}} x\sin x\,\mathrm{d}x = \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x$	M1	
	$= \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$	A1	
	$n = 3 \Rightarrow I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3\times 2\times 1 = \frac{3}{4}\pi^2 - 6$	B1 FT	
	Total:	**3**	

Question	Answer	Marks	Guidance
7	$2\sum\alpha\beta = 1-29 \Rightarrow \sum\alpha\beta = -14$	M1A1	
	$\frac{\sum\alpha\beta}{\alpha\beta\gamma} = \frac{-14}{\alpha\beta\gamma} = -1 \Rightarrow \alpha\beta\gamma = 14$	M1A1 FT	
	$\Rightarrow x^3 + x^2 - 14x - 14 = 0$	A1	
	$\Rightarrow (x+1)(x^2 - 14)$	M1A1	Attempt to factorise cubic
	⇒Solution is –1, in $\pm\sqrt{14}$ any order. Accept ±3.74 (awrt) SR **B1** for correct roots without working	A1	
	Total:	**8**	
8(i)	$z - z^{-1} = \cos\theta + \mathrm{i}\sin\theta - \cos(-\theta) - \mathrm{i}\sin(-\theta) = 2\mathrm{i}\sin\theta$	B1	
	$\left(z-\frac{1}{z}\right)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$	M1A1	
	$\Rightarrow 32\sin^5\theta\mathrm{i} = 2\mathrm{i}\ \sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$	M1A1	Grouping not required at this stage.
	$\Rightarrow 16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$	A1	**M1** for grouping and applying initial result
	Total:	**6**	

Question	Answer	Marks	Guidance
8(ii)	$\int_0^{\frac{1}{3}\pi} 16\sin^5\theta\,\mathrm{d}\theta = \left[-\frac{\cos 5\theta}{5} + \frac{5\cos 3\theta}{3} - 10\cos\theta\right]_0^{\frac{1}{3}\pi}$	**M1A1FT**	
	$= \left[-\frac{1}{10} - \frac{5}{3} - 5\right] - \left[-\frac{1}{5} + \frac{5}{3} - 10\right] = \frac{53}{30}$	**A1**	
	Total:	**3**	
9(i)	$x = 1$	**B1**	
	$y = 2 - x$	**M1A1**	
	Total:	**3**	
9(ii)	$y' = -1 + 4(1-x)^{-2} = 0$	**M1**	
	$\Rightarrow x = -1, 3$	**A1**	
	Turning points are $(-1,5)$ and $(3,-3)$	**A1**	
	Total:	**3**	
9(iii)	(0 , 6)	**B1**	
	$y = 0 \Rightarrow x^2 - 3x + 6$; $\Delta = 9 - 24 \Rightarrow$ No intersection with x-axis	**B1**	
	Total:	**2**	

Question	Answer	Marks	Guidance
9(iv)		B1 FT	Asymptotes correct.
		B1B1	Each branch.
	Total:	3	
10(i)	$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=2t^{\frac{1}{2}}\frac{dy}{dt}$ (AG)	B1	
	$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx}=\left\{t^{-\frac{1}{2}}\frac{dy}{dt}+2t^{\frac{1}{2}}\frac{d^2y}{dt^2}\right\}2t^{\frac{1}{2}}=2\frac{dy}{dt}+4t\frac{d^2y}{dt^2}$ (AG)	M1A1	
	Total:	3	
10(ii)	Substitute in (*): $2\frac{dy}{dt}+4t\frac{d^2y}{dt^2}-16t\frac{dy}{dt}-2\frac{dy}{dt}+12ty=4te^{-t}$ $\Rightarrow\frac{d^2y}{dt^2}-4\frac{dy}{dt}+3y=e^{-t}$ (AG)	B1	
	Total:	1	

Question	Answer	Marks	Guidance
10(iii)	CF: $(m-1)(m-3)=0$	**M1**	
	$y = Ae^{t} + Be^{3t}$	**A1**	
	PI: $y = ke^{-t} \Rightarrow y' = -ke^{-t} \Rightarrow y'' = ke^{-t}$	**M1**	
	$ke^{-t} + 4ke^{-t} + 3ke^{-t} = e^{-t}$	**M1**	
	$\Rightarrow k - \frac{1}{8}$	**A1**	
	GS: $y = Ae^{t} + Be^{3t} + \frac{1}{8}e^{-t}$	**A1 FT**	
	$y = Ae^{x^2} + Be^{3x^2} + \frac{1}{8}e^{-x^2}$	**A1 FT**	
	Total:	**7**	
11(i)		**B1**	Sketch of a cardioid
		B1	Correct orientation and labelled
	Total:	**2**	

Question	Answer	Marks	Guidance
11(ii)	$A=\frac{a^2}{2}\int_{-\pi}^{\pi}\left(1+2\sin\theta+\sin^2\theta\right)\mathrm{d}\theta$	M1	
	$=\left(\frac{a^2}{2}\right)\int_{-\pi}^{\pi}\left(1+2\sin\theta+\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$	M1	
	$=\left(\frac{a^2}{2}\right)\left[\frac{3\theta}{2}-2\cos\theta-\frac{1}{4}\sin 2\theta\right]_{-\pi}^{\pi}=\frac{3\pi a^2}{2}$	M1A1	
	Total:	4	
11(iii)	Show that when r=0, $\theta=-\frac{\pi}{2}$, when r=2a $\theta=\frac{\pi}{2}$ and that $\frac{\mathrm{d}r}{\mathrm{d}\theta}=a\cos\theta$	B1	
	$s=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{a^2\left(1+2\sin\theta+a^2\sin^2\theta\right)+a^2\cos^2\theta}\ \mathrm{d}\theta=\sqrt{2a}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sqrt{1+\sin\theta}\ \mathrm{d}\theta$	M1A1	Uses correct formula for arc length; AG
	Total:	3	
11(iv)	$u=1+\sin\theta\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}\theta}=\cos\theta=\sqrt{1-(u-1)^2}=\sqrt{2u-u^2}$	M1	
	so $s=\sqrt{2a}\int_0^2\frac{1}{\sqrt{2-u}}\mathrm{d}u$ AG	A1	Including limits
	$=\sqrt{2a}\left[-2(2-u)^{\frac{1}{2}}\right]_0^2=4a$	M1A1	
	Total:	4	

Question	Answer	Marks	Guidance
12E(i)	$\int y\mathrm{d}x = \frac{1}{2}\int_0^4 \left(\mathrm{e}^x + \mathrm{e}^{-x}\right)\mathrm{d}x = \frac{1}{2}\left[\mathrm{e}^x - \mathrm{e}^{-x}\right]_0^4 = \frac{1}{2}\left(e^4 - e^{-4}\right)$	M1A1	
	$\int xy\mathrm{d}x = \frac{1}{2}\int_0^4 x\left(e^x + e^{-x}\right)\mathrm{d}x$	M1	
	$\frac{1}{2}\left[\mathrm{e}^x(x-1) - \mathrm{e}^{-x}(x+1)\right]_0^4 = \frac{1}{2}\left(3\mathrm{e}^4 + 2 - 5\mathrm{e}^{-4}\right)$	A1 A1	
	$\frac{1}{2}\int y^2\mathrm{d}x = \frac{1}{8}\int_0^4\left(\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}\right)\mathrm{d}x$	M1	
	$\frac{1}{8}\left[\frac{1}{2}\mathrm{e}^{2x} + 2x + -\frac{1}{2}\mathrm{e}^{-2x}\right]_0^4 = \frac{1}{16}\left(e^8 + 16 - \mathrm{e}^{-8}\right)$	A1 A1	Uses correct formulae for $\overline{x}$ and $\overline{y}$
	$\overline{x} = \frac{3\mathrm{e}^4 + 2 - 5\mathrm{e}^{-4}}{\mathrm{e}^4 - \mathrm{e}^{-4}}\left(= \frac{3\mathrm{e}^2 + 5\mathrm{e}^{-2}}{\mathrm{e}^2 + \mathrm{e}^{-2}}\right)$; $\overline{y} = \frac{1}{8}\left(\frac{\mathrm{e}^8 + 16 - \mathrm{e}^{-8}}{\mathrm{e}^4 - \mathrm{e}^{-4}}\right)$ (OE)	M1 A1	
	Total:	**10**	
12E(ii)	$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + (y')^2} = \sqrt{\frac{1}{4}\left(\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}\right)} = \sqrt{\frac{1}{4}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)^2} = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)$ (AG)	B1	
	$S = 2\pi\int_0^4 \frac{1}{4}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)^2 \mathrm{d}x \quad = \frac{\pi}{2}\int_0^4\left(\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}\right)\mathrm{d}x$	M1 A1	
	$= \frac{\pi}{2}\left[\frac{\mathrm{e}^{2x}}{2} + 2x - \frac{\mathrm{e}^{-2x}}{2}\right]_0^4 = \frac{\pi}{2}\left(\frac{\mathrm{e}^8}{2} + 8 - \frac{\mathrm{e}^{-8}}{2}\right)$	A1	
	Total:	**4**	

Question	Answer	Marks	Guidance
12O(i)	$\overrightarrow{AC}=\begin{pmatrix}2\\-2\\-2\end{pmatrix}$ $\quad\overrightarrow{AB}=\begin{pmatrix}2\\-2\\2\end{pmatrix}$ $\quad\overrightarrow{CD}=\begin{pmatrix}2\\-4\\\alpha-1\end{pmatrix}$	B1	May find $\overrightarrow{AD}, \overrightarrow{BC}$ or $\overrightarrow{BD}$ instead of AC
	$\overrightarrow{AB}\times\overrightarrow{CD}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-1&1\\2&-4&\alpha-1\end{vmatrix}=\begin{pmatrix}5-\alpha\\3-\alpha\\-2\end{pmatrix}\sim\begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}$	M1A1	
	$\left\lvert\dfrac{\begin{pmatrix}2\\-2\\-2\end{pmatrix}\begin{pmatrix}\alpha-5\\\alpha-3\\2\end{pmatrix}}{\sqrt{(\alpha-5)^2+(\alpha-3)^2+4}}\right\rvert=2\sqrt{2}\Rightarrow 2\sqrt{2}\sqrt{2\alpha^2-16\alpha+38}=8$	M1A1	Substitutes their vectors into correct formula
	$\Rightarrow 2\alpha^2-16\alpha+38=8\Rightarrow\alpha^2-8\alpha+15=0$	A1	
	$\Rightarrow(\alpha-3)(\alpha-5)=0\Rightarrow\alpha=3$ or 5 (AG)	A1	
	Total:	7	

Question	Answer	Marks	Guidance
12O(ii)	$\overrightarrow{AD} = \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix} \quad \left(or\ \overrightarrow{CD} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \right)$	B1	*Alt method:* Let P be point on AC with parameter $\boldsymbol{\lambda}$ **M1**
	Distance of D from $AC = \dfrac{1}{\sqrt{1+1+1}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -6 & 0 \\ 1 & -1 & -1 \end{vmatrix} = \sqrt{\frac{56}{3}} = 4.32$ (or with CD)	M1A1	Use DP.AC = 0 to find $\boldsymbol{\lambda}$ (= -5/3) **M1** Find length **A1**
	Total:	**3**	
12O(iii)	$ABC\text{: } \mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad ABD\text{: } \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & -3 & 0 \end{vmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$	B1B1	
	$\cos\theta = \left\lvert \frac{-3-2+0}{\sqrt{2}\sqrt{14}} \right\rvert \Rightarrow \theta = 19.1°\ or\ 0.333\ rads$	M1A1	
	Total:	**4**	