FURTHER MATHEMATICS

Paper 9231/11
Paper 11

Key Messages

- Candidates should use correct mathematical notation and language. They should show all the steps in their working, particularly when they are working towards a given solution.
- Candidates should use the methods indicated in the question, and give their answers in the form required.
- Candidates should sketch graphs carefully showing any symmetries, and labelling key points clearly.

General Comments

Candidates had time to tackle the whole paper, though relatively few attempted both alternatives for the final question. In general the standard of their solutions was very high, with work carefully structured and well presented. Calculus and algebraic handling were generally very good, and most candidates were accurate in their work throughout. Basic errors, for example in expanding brackets, can be costly at this level, so candidates need to check their work carefully. Most candidates read the questions carefully so that they could show the methods and answer formats required, and completed all parts of every question. The strongest candidates ensured that their sketch graphs were labelled clearly.

Question 1

Better candidates had no trouble with this question, showing understanding of the given summation. They were able to use the formula to find the required sum in **(i)** and the r^{th} term in **(ii)**. They were also careful to use the correct variables. Different methods were used in **(ii)**. Some candidates worked out the first four terms and from these the general rule, some candidates made a direct comparison with the sum of squares and others used the difference between the r^{th} sum and the $(r-1)^{\text{th}}$ sum. The latter was probably the most efficient method.

Answers: **(i)** $14n^3 + 9n^2$ **(ii)** $6r^2 - 1$

Question 2

Stronger candidates structured their induction proof well, and used correct language to introduce and describe each step. They defined the proposition, or explained it in words at each stage. Many realised that it was not enough just to show that the difference between successive terms was divisible by 4. It is important that candidates truly understand the steps of such a proof, and confirm in clear language what is being done at each stage. Better proofs picked out the inductive step explicitly.

Question 3

In **(i)**, most students used implicit differentiation as required and deduced the given relationship, rearranging expressions if necessary. The standard of differentiation was generally very good. Better candidates were careful in **(ii)** to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and then square it, as required by the given formula.

Answer: **(ii)** $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{2}$

Question 4

The first part of this question was generally very well done, with most candidates finding the determinant accurately, others using reduced row echelon form to identify where the system did not have a unique solution. Good candidates were also able to solve the system of equations without errors in **(ii)**.

Answer: **(i)** $k = 8$. **(ii)** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{-2} \\ \frac{3}{2} \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

Question 5

Most candidates were able to find the three eigenvectors efficiently in **(i)**, and go on to find the eigenvalues of the related matrix in **(ii)**, although some surprising arithmetic errors prevented some otherwise strong candidates from getting full marks.

Answer: **(i)** **i** + 2**j** + 2**k**, **i** + **k** and **j** + **k** for λ = 1, –1 and –2 **(ii)** **i** + 2**j** + 2**k**, **i** + **k** and **j** + **k** for λ = –1, –3 and –4.

Question 6

In general candidates handled the reduction formula in **(i)** very well, and integrated by parts to prove the given result. Some attempted to find $\mathbf{I_1}$ by using the formula but it had only been proven for n $\geqslant$ 2. Most used integration by parts to find $\mathbf{I_1}$, and went on to substitute into the reduction formula correctly in **(ii)**.

Answer: **(ii)** $\frac{3}{4}\pi^2 - 6$

Question 7

Candidates were usually able to find the cubic equation from the given equations, though there were a few sign errors. A number of candidates, including some strong ones, stopped after this point and seemed to have forgotten that they were required to find the roots. Those who did go on were mostly successful in factorising the cubic as $(x+1)(x^2-14)$ and therefore identifying the three roots.

Answer: $x^3 + x^2 - 14x - 14 = 0$, $x = -1$, $\pm\sqrt{14}$

Question 8

Most candidates proved the given identity, some by going back to basics, some by applying de Moivre's rule. Unfortunately, some candidates overlooked this preliminary question and went straight to the main part. Using the method implied in the question usually led to full marks. Most candidates were careful in their handling of the constants, and in using the correct substitutions including the factor 2i. Those candidates who did not follow the correct method struggled to manipulate their trigonometric expressions. The integration required for **(ii)** was very well done, with most candidates giving their answer in exact form as required in the question.

Answer: **(ii)** $\frac{53}{30}$

Question 9

Candidates were able to state the vertical asymptote and find the oblique asymptote with very few errors in **(i)**, and differentiate to find the turning points in **(ii)**. Stronger candidates calculated the determinant or completed the square to show that the curve did not intersect the *x*-axis, and almost all found the *y* intercept. The graph sketching was generally well done in this question, with lines carefully drawn and labelled, and key points also clearly labelled.

Answers: **(i)** $x = 1$ $y = 2 - x$ **(ii)** $(-1, 5)$ and $(3, -3)$ **(iii)** $(0, 6)$

Question 10

Stronger candidates were clear in their method and used correct notation to show how they were applying the chain rule at each stage. It is very important that each step is shown when working towards a given answer. Some candidates lost marks through missing vital working, or using the differential notation badly. Most candidates knew how to solve the differential equation in **(iii)**. There were a few sign errors but most knew the correct format for the complementary function and the particular integral though some used the wrong variables. Careful candidates remembered to give their final answer in terms of *x* as required in the question.

Answer: **(iii)** $y = Ae^{x^2} + Be^{3x^2} + \frac{1}{8}e^{-x^2}$

Question 11

Good sketches were symmetrical and well labelled in terms of the parameter, *a*. Most candidates drew a cardioid, though some did not complete the section below the initial line. Most candidates were able to find the area enclosed by *C* using a double angle substitution correctly, being careful with the constants as well as the integration itself. Although many remembered the formula for finding the arc length, not all justified the limits as well as the integrand. The strongest candidates were able to complete all the elements of the substitution, though some did not go on to perform the relatively simple final integration. Those who did were usually accurate.

Answers: **(ii)** $\frac{3\pi a^2}{2}$ **(iv)** $4a$

Question 12

EITHER:

Candidates who chose this option usually knew the formulae for the centroid, and were able to make a good attempt at the integration required, though sign errors (or occasionally putting $e^0 = 0$) led to some wrong final answers. In **(ii)**, good candidates showed the necessary working to justify the given expression for $\frac{\mathrm{d}s}{\mathrm{d}x}$ and knew the formula for the surface area so could complete **(ii)** correctly.

Answer: **(i)** $\bar{x} = \frac{3e^4 + 2 - 5e^{-4}}{e^4 - e^{-4}} \left(= \frac{3e^2 + 5e^{-2}}{e^2 + e^{-2}} \right)$; $\bar{y} = \frac{1}{8}\left(\frac{e^8 + 16 - e^{-8}}{e^4 - e^{-4}} \right)$ **(ii)** S = $\frac{\pi}{2}\left(\frac{e^8}{2} + 8 - \frac{e^{-8}}{2} \right)$

OR

Most candidates who chose this option were able to solve the first part of the question by finding the cross product correctly and then substituting into the standard formula. A small number attempted to find the coordinates of the end points of the line of shortest distance, but found it difficult to handle three unknowns. Others found the direction vector of this line, and assumed that this was the vector representing the shortest length itself. There were various ways of completing the second part too, with the most common method being to use the cross product formula, but a number of students found the coordinates of the foot of the perpendicular and hence its length. The final part was very well done with candidates able to find the two normal vectors and then using the scalar product to find the angle between the planes. Most candidates identified the acute angle as required.

Answer: **(ii)** $\sqrt{\frac{56}{3}}$ or 4.32 **(iii)** $\theta = 19.1°$ or 0.333 rads

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

Key Messages

- Candidates should use correct mathematical notation and language. They should show all the steps in their working, particularly when they are working towards a given solution.
- Candidates should use the methods indicated in the question, and give their answers in the form required.
- Candidates should sketch graphs carefully showing any symmetries, and labelling key points clearly.

General Comments

Candidates had time to tackle the whole paper, though relatively few attempted both alternatives for the final question. In general the standard of their solutions was very high, with work carefully structured and well presented. Calculus and algebraic handling were generally very good, and most candidates were accurate in their work throughout. Basic errors, for example in expanding brackets, can be costly at this level, so candidates need to check their work carefully. Most candidates read the questions carefully so that they could show the methods and answer formats required, and completed all parts of every question. The strongest candidates ensured that their sketch graphs were labelled clearly.

Question 1

Better candidates had no trouble with this question, showing understanding of the given summation. They were able to use the formula to find the required sum in **(i)** and the r^{th} term in **(ii)**. They were also careful to use the correct variables. Different methods were used in **(ii)**. Some candidates worked out the first four terms and from these the general rule, some candidates made a direct comparison with the sum of squares and others used the difference between the r^{th} sum and the $(r-1)^{\text{th}}$ sum. The latter was probably the most efficient method.

Answers: **(i)** $14n^3 + 9n^2$ **(ii)** $6r^2 - 1$

Question 2

Stronger candidates structured their induction proof well, and used correct language to introduce and describe each step. They defined the proposition, or explained it in words at each stage. Many realised that it was not enough just to show that the difference between successive terms was divisible by 4. It is important that candidates truly understand the steps of such a proof, and confirm in clear language what is being done at each stage. Better proofs picked out the inductive step explicitly.

Question 3

In **(i)**, most students used implicit differentiation as required and deduced the given relationship, rearranging expressions if necessary. The standard of differentiation was generally very good. Better candidates were careful in **(ii)** to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and then square it, as required by the given formula.

Answer: **(ii)** $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{2}$

Question 4

The first part of this question was generally very well done, with most candidates finding the determinant accurately, others using reduced row echelon form to identify where the system did not have a unique solution. Good candidates were also able to solve the system of equations without errors in **(ii)**.

Answer: **(i)** $k = 8$. **(ii)** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{-2} \\ \frac{3}{2} \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

Question 5

Most candidates were able to find the three eigenvectors efficiently in **(i)**, and go on to find the eigenvalues of the related matrix in **(ii)**, although some surprising arithmetic errors prevented some otherwise strong candidates from getting full marks.

Answer: **(i)** **i** + 2**j** + 2**k**, **i** + **k** and **j** + **k** for λ = 1, –1 and –2 **(ii)** **i** + 2**j** + 2**k**, **i** + **k** and **j** + **k** for λ = –1, –3 and –4.

Question 6

In general candidates handled the reduction formula in **(i)** very well, and integrated by parts to prove the given result. Some attempted to find $\mathbf{I_1}$ by using the formula but it had only been proven for n $\geqslant$ 2. Most used integration by parts to find $\mathbf{I_1}$, and went on to substitute into the reduction formula correctly in **(ii)**.

Answer: **(ii)** $\frac{3}{4}\pi^2 - 6$

Question 7

Candidates were usually able to find the cubic equation from the given equations, though there were a few sign errors. A number of candidates, including some strong ones, stopped after this point and seemed to have forgotten that they were required to find the roots. Those who did go on were mostly successful in factorising the cubic as $(x+1)(x^2-14)$ and therefore identifying the three roots.

Answer: $x^3 + x^2 - 14x - 14 = 0$, $x = -1, \pm\sqrt{14}$

Question 8

Most candidates proved the given identity, some by going back to basics, some by applying de Moivre’s rule. Unfortunately, some candidates overlooked this preliminary question and went straight to the main part. Using the method implied in the question usually led to full marks. Most candidates were careful in their handling of the constants, and in using the correct substitutions including the factor 2i. Those candidates who did not follow the correct method struggled to manipulate their trigonometric expressions. The integration required for **(ii)** was very well done, with most candidates giving their answer in exact form as required in the question.

Answer: **(ii)** $\frac{53}{30}$

Question 9

Candidates were able to state the vertical asymptote and find the oblique asymptote with very few errors in **(i)**, and differentiate to find the turning points in **(ii)**. Stronger candidates calculated the determinant or completed the square to show that the curve did not intersect the *x*-axis, and almost all found the *y* intercept. The graph sketching was generally well done in this question, with lines carefully drawn and labelled, and key points also clearly labelled.

Answers: **(i)** $x = 1$ $y = 2 - x$ **(ii)** $(-1, 5)$ and $(3, -3)$ **(iii)** (0, 6)

Question 10

Stronger candidates were clear in their method and used correct notation to show how they were applying the chain rule at each stage. It is very important that each step is shown when working towards a given answer. Some candidates lost marks through missing vital working, or using the differential notation badly. Most candidates knew how to solve the differential equation in **(iii)**. There were a few sign errors but most knew the correct format for the complementary function and the particular integral though some used the wrong variables. Careful candidates remembered to give their final answer in terms of *x* as required in the question.

Answer: **(iii)** $y = Ae^{x^2} + Be^{3x^2} + \frac{1}{8}e^{-x^2}$

Question 11

Good sketches were symmetrical and well labelled in terms of the parameter, *a*. Most candidates drew a cardioid, though some did not complete the section below the initial line. Most candidates were able to find the area enclosed by *C* using a double angle substitution correctly, being careful with the constants as well as the integration itself. Although many remembered the formula for finding the arc length, not all justified the limits as well as the integrand. The strongest candidates were able to complete all the elements of the substitution, though some did not go on to perform the relatively simple final integration. Those who did were usually accurate.

Answers: **(ii)** $\frac{3\pi a^2}{2}$ **(iv)** $4a$

Question 12

EITHER:

Candidates who chose this option usually knew the formulae for the centroid, and were able to make a good attempt at the integration required, though sign errors (or occasionally putting $e^0 = 0$) led to some wrong final answers. In **(ii)**, good candidates showed the necessary working to justify the given expression for $\frac{\mathrm{d}s}{\mathrm{d}x}$ and knew the formula for the surface area so could complete **(ii)** correctly.

Answer: **(i)** $\bar{x} = \frac{3e^4 + 2 - 5e^{-4}}{e^4 - e^{-4}} \left(= \frac{3e^2 + 5e^{-2}}{e^2 + e^{-2}} \right)$; $\bar{y} = \frac{1}{8}\left(\frac{e^8 + 16 - e^{-8}}{e^4 - e^{-4}} \right)$ **(ii)** S = $\frac{\pi}{2}\left(\frac{e^8}{2} + 8 - \frac{e^{-8}}{2} \right)$

OR

Most candidates who chose this option were able to solve the first part of the question by finding the cross product correctly and then substituting into the standard formula. A small number attempted to find the coordinates of the end points of the line of shortest distance, but found it difficult to handle three unknowns. Others found the direction vector of this line, and assumed that this was the vector representing the shortest length itself. There were various ways of completing the second part too, with the most common method being to use the cross product formula, but a number of students found the coordinates of the foot of the perpendicular and hence its length. The final part was very well done with candidates able to find the two normal vectors and then using the scalar product to find the angle between the planes. Most candidates identified the acute angle as required.

Answer: (ii) $\sqrt{\frac{56}{3}}$ or 4.32 **(iii)** $\theta = 19.1°$ or 0.333 rads

FURTHER MATHEMATICS

Paper 9231/13
Paper 13

Key messages

- Candidates should read the questions carefully, and use the required methods. They should take care to present answers in the form specified.
- Candidates should present their solutions in a clear, logical and ordered way.
- Candidates should make sure that they show full working particularly when tackling questions with given answers.

General comments

Most candidates completed the paper, and a small number attempted both options for the final question. Many candidates reached a very high standard, showing excellent communication skills in producing detailed solutions. Calculus and algebraic handling skills were impressive in general, and the majority of candidates showed very good knowledge across the whole syllabus.

Question 1

The majority of candidates used the correct method and were able to substitute for *x* as requested. Good candidates rearranged the equation correctly before squaring so that they could eliminate the fractional powers and reach a cubic equation. Successful candidates followed the instruction (hence) and used their cubic to answer **(ii)**. Some opted to use the properties of roots to answer the final part.

Answers: **(i)** $9y^3 - 12y^2 + 4y - 1 = 0$ **(ii)** $\frac{12}{9}$ or $\frac{4}{3}$ **(iii)** $\frac{4}{9}$

Question 2

Most candidates had the algebraic skills needed to verify the given relationship in **(i)**, writing down all the steps. Working on the right hand side was most successful, and algebra was made easier for those who took out the common factor. Some candidates tried to use partial fractions but this made the manipulation hard to resolve completely. Stronger candidates showed the method of differences clearly to prove the summation in **(ii)** and were able to deduce the sum to infinity by a clear algebraic method to complete **(iii)**.

Answer: **(iii)** $S\infty = 1\frac{1}{4}$

Question 3

Although most candidates knew the essential steps of an induction proof, only the better candidates were able to structure and word their proofs correctly. The algebraic handling of logarithms was very well done and good answers showed when candidates really understood the implications of the inductive step. The strongest candidates gave a full concluding statement.

Question 4

This question was very well done by most candidates. They showed a good grasp of implicit differentiation and where they chose to rearrange their expression for $\frac{dy}{dx}$ they also used the quotient rule accurately. Most candidates were accurate in their handling of the necessary algebra.

Answer: $\frac{d^2y}{dx^2} = \frac{9}{8}$

Question 5

Most candidates found correct expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and substituted them into the formula for the arc length, and many were able to manipulate the resulting expression correctly so that they could find its square root. This enabled them to complete both parts of the question successfully.

Answers: **(i)** $\frac{72}{5}$ **(ii)** 190π

Question 6

Good candidates reversed the differentiation correctly and spotted that they could rewrite x^2 as $(4 + x^2 - 4)$ (or use an equivalent manipulation). This quickly led them to the reduction formulae. Others spent some time trying further integration by parts, or jumped to the answer without showing sufficient working to justify their solution. The second part was successfully completed by most candidates who were able to apply the given formula for the first integration, and then use their answer in the given reduction formula.

Question 7

Most candidates were able to use de Moivre's Theorem and equate real and imaginary parts to find $\tan(4\theta)$ in **(i)**, though a number did not show sufficient working or did not deal correctly with the factor of i. A small number tried to use the expansion of $(z + 1/z)^4$ and $(z - 1/z)^4$ but then encountered difficulties with substituting. In **(ii)** it was pleasing that many candidates related the necessary quartic equation to the trigonometric equation from **(i)**. Stronger candidates then solved the trigonometric equation and knew that four different answers were required to solve fully the quartic equation. These candidates usually gave their answers in the required form by either listing their solutions or stating the correct values for their parameter in a general solution. A small number of candidates forgot to specify values for the parameter.

Answer: **(ii)** $t = \tan\frac{3\pi}{16}$, $\tan\frac{7\pi}{16}$, $\tan\frac{11\pi}{16}$, $\tan\frac{15\pi}{16}$ or equivalent.

Question 8

The majority of candidates remembered the correct form for the complementary function when the auxiliary equation has a repeated root. Most were able to find the particular integral, but some errors occurred when differentiating to find the missing constants. It is very important that candidates use the correct variables at each stage.

Answer: $x = 2e^{-3t} + 8te^{-3t} + 2t^2 - 2t + 1$

Question 9

This question was very well done by candidates, with few errors seen in any part. Candidates were able to calculate vector products correctly to find the normal to the planes and use them to find the equations in **(i)** and **(ii)**. They then used the normal vectors to find the angle between the two planes, and most gave the acute angle as required.

Answers: **(i)** $7x + y - 3z = 6$ **(ii)** $5x - y + 7z = 20$ **(iii)** $\theta = 78.7°$ or 1.37 rad

Question 10

Most candidates completed the first three parts of this question very accurately and efficiently, with the cross product method to find the eigenvectors proving more effective than row operations. Only stronger candidates were able to find the inverse of P and then to multiply out to find the matrix A^n in terms of n. Brackets around the negative values in P and D were essential. Several candidates did not multiply out to get the final answer.

Answers: **(i)** $\lambda_1 = -2$ **(ii)** $\mathbf{e}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ **(iii)** $\lambda_3 = 5 \Rightarrow \mathbf{e}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(iv) $\begin{pmatrix} \left[5^n + (-1)^n - (-2)^n\right] & \left[2.(-2)^n + (-1)^{n+1} - 5^n\right] & \left[5^n - (-2)^n\right] \\ \left[5^n - (-2)^n\right] & \left[2.(-2)^n - 5^n\right] & \left[5^n - (-2)^n\right] \\ \left[5^n - (-1)^n\right] & \left[(-1)^n - 5^n\right] & \left[5^n\right] \end{pmatrix}$

Question 11

EITHER: The first part of **(i)** was well answered either by using a compound angle formula or by reference to the graphs. Whilst most candidates drew the correct loops, many added sections of the graph in the first and third quadrant, forgetting the convention that $r \geqslant 0$. Candidates were able to substitute for sin (2θ) and r to get the required Cartesian equation in **(ii)**. Almost all knew the formula for the area inside a polar curve – the main difficulties were with the limits and slips with a^2 and the constant after a double angle substitution. Those candidates who differentiated $y = r\sin(\theta)$ and set their answer to zero were able to reach the required relationship in the final part. Others attempted to differentiate the Cartesian equation but found it difficult to finish the question.

Answer: **(iii)** $\frac{1}{2}\pi a^2$

OR: This was the more popular choice for **Question 11**. The majority of candidates were able to reduce the matrix and identify the rank, though a small number forgot to answer this part of the question. Most used the resulting matrix to form equations and find the basis for the null space to compare with the given answer. Where candidates decided to check the given vectors did form a basis for the null space, not all remembered to check that the vectors were in the null space as well as linearly independent. Strong candidates showed their understanding of the null space in answering **(ii)**. Most candidates were able to calculate p and q in **(iii)**, but few remembered to check for consistency. The final part was very well answered.

Answers: **(iii)** $p = 2, q = -1$ **(iv)** $\mathbf{x} = \begin{pmatrix} 4 \\ 9 \\ 1 \\ 4 \end{pmatrix}$

FURTHER MATHEMATICS

Paper 9231/21
Paper 21

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a suitable diagram. Annotating a diagram printed in the question paper may be sufficient if the result is clear, but sometimes it will be preferable to draw a fresh diagram within the answer.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. The overall performance seemed a little higher in Statistics than in Mechanics. In the only question which offered a choice, namely **Question 11**, there was certainly a very strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts at the first part. Indeed all questions were answered well by some candidates, with **Questions 2** and **4** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many, but not all. When relevant in Mechanics questions it is helpful to show on a diagram what forces are acting and also their directions as in **Question 2** and the directions of motion of particles, as in **Question 3**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 7, 9** and **10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to three significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances, for example.

Comments on specific questions

Question 1

Although various approaches to finding the exit speed v are possible, a straightforward one is to equate the change in momentum, $0{\cdot}08\times(300-v)$, to the impulse causing it, $1000\times0{\cdot}02$. Many candidates made a good attempt, but care must be taken over the signs, to avoid the implicit assumption that the bullet reverses direction, for example.

Answer: $50\ \mathrm{m\,s^{-1}}$.

Question 2

Candidates are well advised to first identify all the forces acting on the rod, preferably showing them on a diagram so that the symbols used to represent them are clear. This process reveals that three possible forces are unknown, namely the reaction R_A and friction F_A at A and the normal reaction R_P at P, the point of contact of the rod and disc. A further unknown is the required value of k, implying at first sight that four independent equations must be formulated in order to find k. One such follows immediately from the information about limiting equilibrium in the question, namely $F_A=R_A/8$. While there are a variety of possible moment and force resolution equations to choose from, many candidates chose to resolve forces vertically and horizontally and to take moments about A. It is convenient in this process to write the equations in terms of an angle such as OAP, and then replace in due course such factors as the sine and cosine of this angle, namely 4/5 and 3/5 respectively. Some candidates wrongly assumed from the diagram that the radius OP is parallel to DA, resulting in the incorrect values $5/\sqrt{41}$ and $4/\sqrt{41}$ being used for the above sine and cosine. As is sometimes the case in such questions, it is possible to reduce the work required by a careful choice of equations. Thus taking moments about P, for example, together with only a resolution of forces along AB is sufficient since the unknown force R_P is not introduced. Finally, virtually all candidates gave no indication that the friction at A may act either upwards or downwards, giving different values for k, but full marks could be earned by a satisfactory consideration of only one case.

Answer: 4/17 or 2.

Question 3

Most candidates correctly formulated and solved two equations for the velocities of the spheres A and B after their first collision by means of conservation of momentum and Newton's restitution equation, and then combined these equations to give the required expressions for the velocities. The collision of B with the wall needs only the restitution equation, while both restitution and conservation of momentum must again be applied to the second collision between the spheres. Combining the resulting equations with the final speed of A taken as zero yields a quadratic equation for the required coefficient of restitution, with the negative root being rejected.

Answers: **(i)** $\frac{1}{4}(3-e)u$, $\frac{3}{4}(1+e)u$ **(ii)** $\frac{3}{5}$.

Question 4

The first part requires the use of standard formulae and the parallel axes theorem to formulate and then sum the individual moments of inertia of the discs and rods. As in all such questions requiring the moment of inertia of an object composed of several parts, candidates should include sufficient details of their solution so as to maximise potential credit. Those who simply write down a sum of several terms with no explanation whatever run considerable risk, since an error in only one term can reduce the intelligibility of their whole process and thereby lose considerable credit. The second part was seemingly found to be more challenging. A surprising number of candidates ignored the instruction to state the change h in the vertical position of the centre of mass of the object, a task which is greatly simplified by invoking symmetry, and which also assists in finding the required angular velocity. Not all candidates realised that this task involves equating the gain in rotational energy of the object to its loss in potential energy, the latter being simply $4\,mgh$. It is of course needlessly time-wasting to calculate this by instead summing the individual losses in potential energy of the various components of the object. Solution of the resulting energy equation then yields the angular velocity.

Answers: **(i)** $81ma^2/2$ **(ii)** $4a$, $(8/9)\sqrt{(g/a)}$.

Question 5

Most candidates verified the given expression for the speed successfully by applying conservation of energy between A and the point when the string first makes contact with the peg at B. The tensions at A and C may be related to the particle's speed at each of these points by using Newton's second law of motion. The speed at A is of course given to be $\sqrt{(ag)}$, but that at C must be found by another application of conservation of energy between either A or B and C. Some candidates overlooked this step, in effect mistakenly assuming that the speed is unaltered between B and C when finding the tension at C. Equating the expressions for the tensions at A and C then gives the required value of $\cos\alpha$. While the general approach was widely understood, minor errors in its execution were not uncommon.

Answer: **(ii)** $\frac{1}{4}$

Question 6

The probability that obtaining a 6 with a die takes no more than four throws was usually found correctly from $1-q^4$ with $q=5/6$. The starting point in the second part is to formulate the inequality $1-q^{N-1}>0{\cdot}95$, solution of which gives $N-1>16{\cdot}4$ and hence the least integral value of N. A common error here was to use q^N in place of q^{N-1}.

Answers: **(i)** 671/1296 or 0·518 **(ii)** 18.

Question 7

Most candidates understood the general approach to performing this test, even if it was not always carried out perfectly. As in all such tests, the hypotheses should be stated in terms of the population mean and not the sample mean. The unbiased estimate 118/45 or 2·622 of the population variance may be used to calculate a t-value of 1·95. Since it is a one-tail test, comparison with the tabulated value of 1·833 leads to acceptance of the alternative hypothesis, and thus the conclusion that the farmer's claim is justified.

Question 8

As well as integrating $\frac{1}{4}(x-1)$ to find the distribution function $\mathrm{F}(x)$ over $2\leqslant x\leqslant 4$, candidates should preferably state the values of F(x) for other values of x. The first step in the next part is to find or state the distribution function G of Y, preferably simplifying it to, say, $(y^{2/3}-1)/8$ and this is then differentiated to give the required probability density function. Apart from the few candidates who confused median and mean, the method for finding the median value m of Y by equating $\mathrm{G}(m)$ to $\frac{1}{2}$ was well known.

Answers: **(i)** 0 ($x<2$), $x^2/8-x/4$ ($2\leqslant x\leqslant 4$), 1 ($x>4$) **(ii)** $1/(12y^{1/3})$ ($1\leqslant y\leqslant 27$), 0 otherwise **(iii)** $5\sqrt{5}$.

Question 9

As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. Most candidates found an unbiased pooled estimate of 0·3782 for the common variance. This enables the value 1·18 to be found for t, and comparison with the tabulated value of 1·746 leads to acceptance of the null hypothesis, and hence a conclusion of there being no difference between the mean masses of fish produced by the two farmers. Since the question states that the two population variances should be assumed to be equal, it is inappropriate to base the test on an estimate of the variance of the combined populations, as some candidates did.

Question 10

The gradient of the required regression line of y on x is readily found using the standard formula, first finding the relevant summations from the information given in the question. Care is needed, however, to avoid finding the gradient of the regression line of x on y by mistake. The final step is to recall that the mean values of x and y satisfy the equation of the regression line. Assuming instead that a randomly-chosen pair of values from the table of data satisfies the equation is of course invalid, though this was sometimes seen. The required value of the correlation coefficient is similarly found using the standard formula. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho \neq 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison with the tabular value 0·805 leads to a conclusion of there being no evidence of non-zero correlation

Answers: **(i)** $y = 7{\cdot}2 - 0{\cdot}3x$ **(ii)** –0·520.

Question 11 (Mechanics)

This optional question was attempted by a very small minority of candidates, and while many of those who did so made good attempts at the first part, the other two parts were clearly found particularly challenging. The first part requires resolution of forces along the plane to find the tension, and then equating this to the expression resulting from Hooke's Law to find the required value of k. After Q replaces P and it is released from rest, Newton's second law of motion can be applied to derive eventually an equation in the form $\mathrm{d}^2x/\mathrm{d}t^2 = -\omega^2 x$ with $\omega^2 = 4g/a$, thus verifying simple harmonic motion. Achieving this standard form of the SHM equation requires x to be the displacement from the new centre of motion, which is of course the equilibrium position for the new particle Q of mass $2m$. Its position may be found directly by a similar process to that used in the first part, or by applying Newton's law at a general point and then deducing what change of dependent variable is needed in order to remove the constant term from the resulting second-order differential equation. The period is found from the standard formula $2\pi/\omega$. Since the centre of motion is found to be $7a/6$ from O and the lowest point of the motion is at E, given to be $5a/4$ from O, it follows that the amplitude x_0 is $a/12$ or equivalently the highest point of the motion is $13a/12$ from O. The least tension may then be found from $kmg(a/12)/a$ and the maximum acceleration from $\omega^2 x_0$.

Answers: **(i)** 8 **(ii)** $7a/6$ from O, $\pi\sqrt{(a/g)}$ **(iii)** $2mg/3$, $g/3$.

Question 11 (Statistics)

Since the first part requires that the mean number of damaged pots per pack is shown to equal the given value, rather than its unknown value be found, it is necessary to demonstrate a full understanding of the process. A very few candidates just wrote down 414/250=1·656, which is insufficient. The required value of a in the second part may be found by evaluating $250 \times {}^6C_2(1-p)^4p^2$ with p equal to one-sixth of the mean, namely 0·276, and similarly for b. A few candidates found a and b by instead taking an appropriate multiple of an adjacent value in the table of expected frequencies, but this did not always yield values correct to two decimal places. In the final part a clear statement of the null hypothesis, such as 'the binomial distribution is a good fit to the data', is preferable to a more vague statement such as 'it is a suitable model'. Candidates should be aware that in such questions cells may need to be combined in order to ensure that all the expected values are at least 5, and here the last three cells must be combined. Apart from this, the goodness of fit test was often carried out well. Comparison of the calculated value 14·9 of χ^2 with the critical value 11·34 leads to rejection of the null hypothesis, and hence the conclusion that the manager's belief is not justified.

Answer: **(ii)** $a = 78{\cdot}49$, $b = 11{\cdot}41$.

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a suitable diagram. Annotating a diagram printed in the question paper may be sufficient if the result is clear, but sometimes it will be preferable to draw a fresh diagram within the answer.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. The overall performance seemed a little higher in Statistics than in Mechanics. In the only question which offered a choice, namely **Question 11**, there was certainly a very strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts at the first part. Indeed all questions were answered well by some candidates, with **Questions 2** and **4** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many, but not all. When relevant in Mechanics questions it is helpful to show on a diagram what forces are acting and also their directions as in **Question 2** and the directions of motion of particles, as in **Question 3**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 7, 9** and **10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to three significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances, for example.

Comments on specific questions

Question 1

Although various approaches to finding the exit speed v are possible, a straightforward one is to equate the change in momentum, $0{\cdot}08\times(300-v)$, to the impulse causing it, $1000\times0{\cdot}02$. Many candidates made a good attempt, but care must be taken over the signs, to avoid the implicit assumption that the bullet reverses direction, for example.

Answer: $50\ \mathrm{m\,s^{-1}}$.

Question 2

Candidates are well advised to first identify all the forces acting on the rod, preferably showing them on a diagram so that the symbols used to represent them are clear. This process reveals that three possible forces are unknown, namely the reaction R_A and friction F_A at A and the normal reaction R_P at P, the point of contact of the rod and disc. A further unknown is the required value of k, implying at first sight that four independent equations must be formulated in order to find k. One such follows immediately from the information about limiting equilibrium in the question, namely $F_A=R_A/8$. While there are a variety of possible moment and force resolution equations to choose from, many candidates chose to resolve forces vertically and horizontally and to take moments about A. It is convenient in this process to write the equations in terms of an angle such as OAP, and then replace in due course such factors as the sine and cosine of this angle, namely 4/5 and 3/5 respectively. Some candidates wrongly assumed from the diagram that the radius OP is parallel to DA, resulting in the incorrect values $5/\surd41$ and $4/\surd41$ being used for the above sine and cosine. As is sometimes the case in such questions, it is possible to reduce the work required by a careful choice of equations. Thus taking moments about P, for example, together with only a resolution of forces along AB is sufficient since the unknown force R_P is not introduced. Finally, virtually all candidates gave no indication that the friction at A may act either upwards or downwards, giving different values for k, but full marks could be earned by a satisfactory consideration of only one case.

Answer: 4/17 or 2.

Question 3

Most candidates correctly formulated and solved two equations for the velocities of the spheres A and B after their first collision by means of conservation of momentum and Newton's restitution equation, and then combined these equations to give the required expressions for the velocities. The collision of B with the wall needs only the restitution equation, while both restitution and conservation of momentum must again be applied to the second collision between the spheres. Combining the resulting equations with the final speed of A taken as zero yields a quadratic equation for the required coefficient of restitution, with the negative root being rejected.

Answers: **(i)** $\frac{1}{4}(3-e)u$, $\frac{3}{4}(1+e)u$ **(ii)** $\frac{3}{5}$.

Question 4

The first part requires the use of standard formulae and the parallel axes theorem to formulate and then sum the individual moments of inertia of the discs and rods. As in all such questions requiring the moment of inertia of an object composed of several parts, candidates should include sufficient details of their solution so as to maximise potential credit. Those who simply write down a sum of several terms with no explanation whatever run considerable risk, since an error in only one term can reduce the intelligibility of their whole process and thereby lose considerable credit. The second part was seemingly found to be more challenging. A surprising number of candidates ignored the instruction to state the change h in the vertical position of the centre of mass of the object, a task which is greatly simplified by invoking symmetry, and which also assists in finding the required angular velocity. Not all candidates realised that this task involves equating the gain in rotational energy of the object to its loss in potential energy, the latter being simply $4\,mgh$. It is of course needlessly time-wasting to calculate this by instead summing the individual losses in potential energy of the various components of the object. Solution of the resulting energy equation then yields the angular velocity.

Answers: **(i)** $81ma^2/2$ **(ii)** $4a$, $(8/9)\surd(g/a)$.

Question 5

Most candidates verified the given expression for the speed successfully by applying conservation of energy between A and the point when the string first makes contact with the peg at B. The tensions at A and C may be related to the particle's speed at each of these points by using Newton's second law of motion. The speed at A is of course given to be $\sqrt{(ag)}$, but that at C must be found by another application of conservation of energy between either A or B and C. Some candidates overlooked this step, in effect mistakenly assuming that the speed is unaltered between B and C when finding the tension at C. Equating the expressions for the tensions at A and C then gives the required value of $\cos \alpha$. While the general approach was widely understood, minor errors in its execution were not uncommon.

Answer: **(ii)** $\frac{1}{4}$

Question 6

The probability that obtaining a 6 with a die takes no more than four throws was usually found correctly from $1–q^4$ with $q=5/6$. The starting point in the second part is to formulate the inequality $1–q^{N–1}>0{\cdot}95$, solution of which gives $N–1>16{\cdot}4$ and hence the least integral value of N. A common error here was to use q^N in place of $q^{N–1}$.

Answers: **(i)** 671/1296 or 0·518 **(ii)** 18.

Question 7

Most candidates understood the general approach to performing this test, even if it was not always carried out perfectly. As in all such tests, the hypotheses should be stated in terms of the population mean and not the sample mean. The unbiased estimate 118/45 or 2·622 of the population variance may be used to calculate a t-value of 1·95. Since it is a one-tail test, comparison with the tabulated value of 1·833 leads to acceptance of the alternative hypothesis, and thus the conclusion that the farmer's claim is justified.

Question 8

As well as integrating $\frac{1}{4}(x–1)$ to find the distribution function $\mathrm{F}(x)$ over $2\leqslant x\leqslant 4$, candidates should preferably state the values of F(x) for other values of x. The first step in the next part is to find or state the distribution function G of Y, preferably simplifying it to, say, $(y^{2/3} – 1)/8$ and this is then differentiated to give the required probability density function. Apart from the few candidates who confused median and mean, the method for finding the median value m of Y by equating $\mathrm{G}(m)$ to $\frac{1}{2}$ was well known.

Answers: **(i)** 0 $(x<2)$, $x^2/8–x/4$ $(2\leqslant x\leqslant 4)$, 1 $(x>4)$ **(ii)** $1/(12y^{1/3})$ $(1\leqslant y\leqslant 27)$, 0 otherwise **(iii)** $5\sqrt{5}$.

Question 9

As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. Most candidates found an unbiased pooled estimate of 0·3782 for the common variance. This enables the value 1·18 to be found for t, and comparison with the tabulated value of 1·746 leads to acceptance of the null hypothesis, and hence a conclusion of there being no difference between the mean masses of fish produced by the two farmers. Since the question states that the two population variances should be assumed to be equal, it is inappropriate to base the test on an estimate of the variance of the combined populations, as some candidates did.

Question 10

The gradient of the required regression line of y on x is readily found using the standard formula, first finding the relevant summations from the information given in the question. Care is needed, however, to avoid finding the gradient of the regression line of x on y by mistake. The final step is to recall that the mean values of x and y satisfy the equation of the regression line. Assuming instead that a randomly-chosen pair of values from the table of data satisfies the equation is of course invalid, though this was sometimes seen. The required value of the correlation coefficient is similarly found using the standard formula. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho \neq 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison with the tabular value 0·805 leads to a conclusion of there being no evidence of non-zero correlation

Answers: **(i)** $y = 7{\cdot}2 - 0{\cdot}3x$ **(ii)** $-0{\cdot}520$.

Question 11 (Mechanics)

This optional question was attempted by a very small minority of candidates, and while many of those who did so made good attempts at the first part, the other two parts were clearly found particularly challenging. The first part requires resolution of forces along the plane to find the tension, and then equating this to the expression resulting from Hooke's Law to find the required value of k. After Q replaces P and it is released from rest, Newton's second law of motion can be applied to derive eventually an equation in the form $d^2x/dt^2 = -\omega^2 x$ with $\omega^2 = 4g/a$, thus verifying simple harmonic motion. Achieving this standard form of the SHM equation requires x to be the displacement from the new centre of motion, which is of course the equilibrium position for the new particle Q of mass $2m$. Its position may be found directly by a similar process to that used in the first part, or by applying Newton's law at a general point and then deducing what change of dependent variable is needed in order to remove the constant term from the resulting second-order differential equation. The period is found from the standard formula $2\pi/\omega$. Since the centre of motion is found to be $7a/6$ from O and the lowest point of the motion is at E, given to be $5a/4$ from O, it follows that the amplitude x_0 is $a/12$ or equivalently the highest point of the motion is $13a/12$ from O. The least tension may then be found from $kmg(a/12)/a$ and the maximum acceleration from $\omega^2 x_0$.

Answers: **(i)** 8 **(ii)** $7a/6$ from O, $\pi\sqrt{(a/g)}$ **(iii)** $2mg/3$, $g/3$.

Question 11 (Statistics)

Since the first part requires that the mean number of damaged pots per pack is shown to equal the given value, rather than its unknown value be found, it is necessary to demonstrate a full understanding of the process. A very few candidates just wrote down 414/250=1·656, which is insufficient. The required value of a in the second part may be found by evaluating $250 \times {}^6C_2(1-p)^4p^2$ with p equal to one-sixth of the mean, namely 0·276, and similarly for b. A few candidates found a and b by instead taking an appropriate multiple of an adjacent value in the table of expected frequencies, but this did not always yield values correct to two decimal places. In the final part a clear statement of the null hypothesis, such as 'the binomial distribution is a good fit to the data', is preferable to a more vague statement such as 'it is a suitable model'. Candidates should be aware that in such questions cells may need to be combined in order to ensure that all the expected values are at least 5, and here the last three cells must be combined. Apart from this, the goodness of fit test was often carried out well. Comparison of the calculated value 14·9 of χ^2 with the critical value 11·34 leads to rejection of the null hypothesis, and hence the conclusion that the manager's belief is not justified.

Answer: **(ii)** $a = 78{\cdot}49$, $b = 11{\cdot}41$.

FURTHER MATHEMATICS

Paper 9231/23
Paper 23

Key messages

To score full marks in the paper, candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a suitable diagram. Annotating a diagram printed in the question paper may be sufficient if the result is clear, but sometimes it will be preferable to draw a fresh diagram within the answer.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Question 1** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many, but not all. When relevant in Mechanics questions it is helpful to show on a diagram what forces are acting and also their directions as in **Question 4** and the directions of motion of particles, as in **Question 3**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 7** and **11** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances, for example.

<u>Comments on specific questions</u>

Question 1

Many candidates found this question challenging. Success requires a clear understanding of whether each of the various forces acts directly on the disc or on the block. One possible approach is to formulate an equation of angular motion for the disc in terms of its angular acceleration under the action of the tension T of the string, together with an equation of motion for the block in terms of its linear acceleration under the action of its weight, the tension, and the resisting force. Relating the linear and angular accelerations then yields the required result for T. Alternatively two energy equations may be formulated in terms of the angular and linear speeds, which are related similarly to the accelerations. It is unwise to represent the linear acceleration by the popular symbol a, since this risks confusion with the radius of the disc, and it should be noted that the mass of the block is here $3m$ and not the usual m.

Answer: $\dfrac{3mg}{10}$

Question 2

Almost all candidates realised that the distances of L and M from the centre of motion O must be in the same given ratio as the magnitudes of their accelerations, from which the distance OM follows immediately. The second part requires significantly more thought, and can be approached in a variety of ways, some based on the standard SHM formula $x = a \sin \omega t$ and others on $x = a \cos \omega t$, with $a = 2{\cdot}5$. In essence the times to travel to L and to M from some point such as O or an outer point of the motion are found, and then related to the given time 2 s. This results in the value $\dfrac{\pi}{4}$ of ω and hence the period using $\dfrac{2\pi}{\omega}$. Solving this part-question by use of the geometrical approach to SHM based on motion round a circle of radius a with uniform angular velocity ω is particularly easy in this case, but this approach was very rarely seen. Finally the required speed of P at L may be found from an SHM formula such as $\omega \sqrt{(a^2 - x^2)}$ with $x = 1{\cdot}5$ or $a\omega \cos \omega t$ with $\omega t = \sin^{-1}(\dfrac{x}{a})$.

Answers: (**i**) 2 m; (**ii**) 8 s; (**iii**) $\dfrac{\pi}{2}$ms^{-1}.

Question 3

Almost all candidates correctly formulated and solved two equations for the speeds of the spheres A and B after their collision by means of conservation of momentum and Newton's restitution equation, and then applied the latter process again for the collision between B and the wall, hence verifying the given value of B's subsequent speed. Candidates found the second part somewhat more challenging, and a variety of different approaches to finding the required distance x were seen. It is possible to write down, for example, a single equation for x by equating the time for A to travel d – x to the sum of the times for B to travel d to the wall and then x back to meet A. Alternatively the distance $\dfrac{d}{5}$ which A has travelled when B collides with the wall may be found, and then the times for the subsequent travel of A and B related. A small minority of candidates based their approach on the ratios of the speeds of the spheres. In all cases, it is helpful to the Examiners, particularly when the required distance is found incorrectly, to give an adequate explanation of the approach being employed.

Answer: (**ii**) $\dfrac{d}{2}$

Question 4

In finding the required tension T of the string, it is essential to take moments for the rod about A in order to relate T and W since resolution of forces will not suffice. Some candidates found separately the moments of the horizontal and vertical components of T but this is an unnecessary complication since the perpendicular distance of OB from A is readily determined to be $3a \sin \theta$. Once the length $\frac{26a}{5}$ of the string and hence its extension $\frac{6a}{5}$ have been found by trigonometry, Hooke's Law may be applied to find the required modulus of elasticity. Not all candidates realised that resolution of forces may be used in the final part to find the horizontal component $T \cos \theta$ and vertical component $W + T \sin \theta$ of the unknown force at A, and hence its required direction.

Answers: **(i)** $\frac{7W}{30}$; **(ii)** $\frac{7W}{9}$; **(iii)** 80·7°.

Question 5

Almost all candidates correctly verified the given equation by conservation of energy between P and Q. The second part requires that the tensions at P and Q be found in terms of u and v respectively, and then twice the former tension equated to the latter one. While many candidates successfully obtained the required expression for u in terms of a and g by eliminating v using the equation from part **(i)**, they often overlooked the requirement in the question to first state explicitly an equation relating u^2 and v^2, thereby needlessly forfeiting credit. The long-standing advice to read the question carefully is still relevant. The least tension occurs of course at the highest point of the motion, and it is found from $\frac{mV^2}{a} - mg$ where the speed V at this point may found from conservation of energy between either P or Q and the highest point.

Answers: **(ii)** $v^2 = 2u^2 - \frac{11ag}{5}$, $\sqrt{(5ag)}$; **(iii)** $\frac{18mg}{5}$

Question 6

Most candidates produced good answers to this question, which requires that the usual expression for the pooled estimate of the common variance be equated to 10, and the resulting quadratic equation solved for the required integer value of N. Equating the estimated variance for X to that for Y, ignoring the given information about the pooled variance, is not what is required here.

Answer: 4.

Question 7

The required value of the correlation coefficient is readily found from the gradients of the two given regression lines, and like all non-exact numerical answers (other than angles in degrees) should be stated correct to 3 significant figures. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho \neq 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison with the tabular value 0·576 leads to a conclusion of there being evidence of non-zero correlation.

Answer: **(i)** 0·654.

Question 8

A key decision to be made is whether to use an estimate of a combined variance or a pooled estimate of a common population variance. In the absence of any information in the question about the variances of the two distributions, it is appropriate to estimate and use a combined variance. However the alternative is acceptable to the Examiners provided candidates state explicitly that they are assuming the two population variances to be equal. Most candidates knew how to find a confidence interval for the difference in the population means, though the final result differs slightly depending on which variance estimate and tabular value are used. The solution given here is for the combined variance and a z-value of 1·96, used by the majority of candidates.

Answer: [4·50, 7·74].

Question 9

Most candidates produced correct answers to the first two parts, finding the value of a by equating the integral of f(x) over the range $x \geqslant 0$ to unity and then recalling the properties of the exponential distribution in order to state the value of E(X). The meaning of interquartile range was less well understood, and even some candidates who found the values of the upper and lower quartiles correctly, namely 2 and 0·415, did not appreciate that the interquartile range means their difference. The usual first step in the final part is to find or state the distribution function of Y, preferably simplifying it to $1 - \frac{1}{y}$, over the range corresponding to $x \geqslant 0$. This is then differentiated to give the required probability density function g(y). Candidates should state that their result holds true for $y \geqslant 1$, with g(y) = 0 otherwise.

Answers: (**i**) ln 2 or 0·693; (**ii**) $\frac{1}{\ln 2}$ or 1·44; (**iii**) 1·58; (**iv**) 0 (y < 1), $\frac{1}{y^2}$ ($y \geqslant 1$).

Question 10

Since the first part requires that the mean number of rooms that are occupied each night is shown to equal the given value, rather than its unknown value be found, it is necessary to demonstrate a full understanding of the process. Simply writing down, for example, $\frac{325}{100}$ = 3·25 is insufficient. Similarly the second part requires a statement that the expected value for x = 3 is found from the expression 100 λ^3 $e^{-\lambda}$/3! with λ = 3·25. The expected values for x = 6 and x = 7 are found in a similar way, and finally that for $x \geqslant 7$ satisfies the total sum of the expected values equalling 100. This final step was occasionally overlooked, leading to inaccurate results later, but on the whole the first two parts were well done. In the final part a clear statement of the null hypothesis, such as 'the Poisson distribution is a good fit to the data', is preferable to a more vague statement such as 'it is a suitable model'. Candidates should be aware that in such questions cells may need to be combined in order to ensure that all the expected values are at least 5, and here the first two cells must be combined, as must also the last two. Apart from this, the goodness of fit test was often carried out well. Comparison of the calculated value 5·00 of χ^2 with the critical value 9·488 leads to acceptance of the null hypothesis, and hence the conclusion that the Poisson distribution is a suitable model.

Answer: (**ii**) 6·35, 4·77.

Question 11 (Mechanics)

This optional question was attempted by just under a third of the candidates, but many of those who did so made good attempts, particularly at the first part. This requires the use of standard formulae and the parallel axes theorem to formulate and then sum the individual moments of inertia of the rod, solid sphere and spherical shell. As in all such questions where a given result must be shown, candidates should include sufficient details of their solution so as to justify full credit. Those who simply write down a sum of several terms with no explanation of any kind run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose considerable credit. In the second part of the question many candidates realised that the gain in rotational energy of the object should be equated to its loss in potential energy when CA becomes vertical, though some found calculation of the latter difficult. Solution of the resulting equation yields the angular speed ω of the object, and the required speed of the centre O is then $2a\omega$.

Answer: (**ii**) $\left(\frac{10}{17}\right)\sqrt{(ag)}$.

Question 11 (Statistics)

The gradient of the required regression line of y on x is readily found using the standard formula since the values of all the relevant summations are given in the question. Care is needed, however, to avoid finding the gradient of the regression line of x on y by mistake. The final step is to recall that the mean values of x and y satisfy the equation of the regression line. Assuming instead that a randomly-chosen pair of values from the table of results satisfies the equation is of course invalid. The question does not explicitly specify the use of a particular test in the second part, but most candidates realised that a paired-sample t-test is appropriate, and conducted the test well. Use of some other test is not acceptable. The first step in the calculation is to find the differences between the pairs of observations and then base the test on them. It is usually essential to retain the signs of the differences and not just consider their magnitudes, but as it happens they are all of the same sign here. The mean of the resulting sample is then 0·275 and the unbiased estimate of the population variance is $\frac{11}{280}$ or 0·0393. This gives a calculated value of t of 1·07, and comparison with the tabulated value 1·415 leads to the conclusion of there being no evidence to support the coach's belief. As in all such tests, candidates should state their hypotheses explicitly in terms of the population rather than the sample means. A further requirement in this case is to make the hypotheses unambiguous, corresponding to the mean times decreasing rather than increasing over the year, for example.

Answer: (**i**) $y = 0{\cdot}782x + 4{\cdot}93$.