Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/13**

Paper 1 **October/November 2015**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

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READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **4** printed pages.

1 The curve C is defined parametrically by

$$x = 2\cos^3 t \quad \text{and} \quad y = 2\sin^3 t, \quad \text{for } 0 < t < \tfrac{1}{2}\pi.$$

Show that, at the point with parameter t,

$$\frac{d^2y}{dx^2} = \tfrac{1}{6}\sec^4 t \operatorname{cosec} t.$$ [4]

2 Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 7 - 2t^2.$$ [6]

3 Given that a is a constant, prove by mathematical induction that, for every positive integer n,

$$\frac{d^n}{dx^n}(xe^{ax}) = na^{n-1}e^{ax} + a^n xe^{ax}.$$ [6]

4 The sequence $a_1, a_2, a_3, \ldots$ is such that, for all positive integers n,

$$a_n = \frac{n+5}{\sqrt{(n^2-n+1)}} - \frac{n+6}{\sqrt{(n^2+n+1)}}.$$

The sum $\sum_{n=1}^{N} a_n$ is denoted by S_N. Find

(i) the value of S_{30} correct to 3 decimal places, [3]

(ii) the least value of N for which $S_N > 4.9$. [4]

5 The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are integers, has roots α, β and γ, such that

$$\alpha + \beta + \gamma = 15,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 83.$$

Write down the value of p and find the value of q. [3]

Given that α, β and γ are all real and that $\alpha\beta + \alpha\gamma = 36$, find α and hence find the value of r. [5]

6 The matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 10 & -7 & 10 \\ 7 & -5 & 8 \end{pmatrix},$$

has eigenvalues 1 and 3. Find corresponding eigenvectors. [3]

It is given that $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$. Find the corresponding eigenvalue. [2]

Find a diagonal matrix $\mathbf{D}$ and matrices $\mathbf{P}$ and $\mathbf{P}^{-1}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. [5]

7 The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & -2 & -3 & 1 \\ 3 & -5 & -7 & 7 \\ 5 & -9 & -13 & 9 \\ 7 & -13 & -19 & 11 \end{pmatrix}.$$

Find the rank of $\mathbf{M}$ and a basis for the null space of T. [6]

The vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ is denoted by $\mathbf{e}$. Show that there is a solution of the equation $\mathbf{Mx} = \mathbf{Me}$ of the form $\mathbf{x} = \begin{pmatrix} a \\ b \\ -1 \\ -1 \end{pmatrix}$, where the constants a and b are to be found. [4]

8 The curve C has equation $y = \dfrac{2x^2 + kx}{x + 1}$, where k is a constant. Find the set of values of k for which C has no stationary points. [5]

For the case $k = 4$, find the equations of the asymptotes of C and sketch C, indicating the coordinates of the points where C intersects the coordinate axes. [6]

9 It is given that $I_n = \displaystyle\int_1^{\mathrm{e}} (\ln x)^n \, \mathrm{d}x$ for $n \geqslant 0$. Show that

$$I_n = (n - 1)[I_{n-2} - I_{n-1}] \quad \text{for } n \geqslant 2. \qquad [6]$$

Hence find, in an exact form, the mean value of $(\ln x)^3$ with respect to x over the interval $1 \leqslant x \leqslant \mathrm{e}$. [6]

10 Using de Moivre's theorem, show that

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}. \qquad [5]$$

Hence show that the equation $x^2 - 10x + 5 = 0$ has roots $\tan^2\left(\frac{1}{5}\pi\right)$ and $\tan^2\left(\frac{2}{5}\pi\right)$. [4]

Deduce a quadratic equation, with integer coefficients, having roots $\sec^2\left(\frac{1}{5}\pi\right)$ and $\sec^2\left(\frac{2}{5}\pi\right)$. [3]

[Question 11 is printed on the next page.]

11 Answer only **one** of the following two alternatives.

EITHER

The points A, B and C have position vectors $\mathbf{i}$, $2\mathbf{j}$ and $4\mathbf{k}$ respectively, relative to an origin O. The point N is the foot of the perpendicular from O to the plane ABC. The point P on the line-segment ON is such that $OP = \frac{3}{4}ON$. The line AP meets the plane OBC at Q. Find a vector perpendicular to the plane ABC and show that the length of ON is $\dfrac{4}{\sqrt{(21)}}$. [4]

Find the position vector of the point Q. [5]

Show that the acute angle between the planes ABC and ABQ is $\cos^{-1}\left(\frac{2}{3}\right)$. [5]

OR

The curve C has polar equation $r = a(1 - \cos\theta)$ for $0 \leqslant \theta < 2\pi$. Sketch C. [2]

Find the area of the region enclosed by the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$, the half-line $\theta = \frac{1}{2}\pi$ and the half-line $\theta = \frac{3}{2}\pi$. [5]

Show that

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\theta}\right)^2 = 4a^2\sin^2\left(\tfrac{1}{2}\theta\right),$$

where s denotes arc length, and find the length of the arc of C for which $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$. [7]