CAMBRIDGE INTERNATIONAL EXAMINATIONS

Cambridge International Advanced Level

MARK SCHEME for the October/November 2015 series

9231 FURTHER MATHEMATICS

9231/23 Paper 2, maximum raw mark 100

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Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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Question Number	Mark Scheme Details			Part Mark	Total
1	Find 3 independent equations for T, R_A, R_B:				
	Resolve horizontally:	$R_B = T \cos \alpha$	M1 A1		
	Resolve vertically:	$R_A = W + T \sin \alpha$	M1 A1		
	Take moments about A: (a may be omitted from moment eqns)	$R_B\, 3a \sin \theta = W(3a/2) \cos \theta$ $+ T a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ *or* $+ T a \sin(\alpha + \theta)$ *or* $+ T\, 3a \cos \theta \sin \alpha$	M1 A1		
	Take moments about B:	$R_A 3a \cos \theta = W(3a/2) \cos \theta$ $+ T\, 2a(\sin \alpha \cos \theta + \cos \alpha \sin \theta)$ *or* $+ T\, 2a \sin(\alpha + \theta)$ *or* $+ T\, 3a \sin \theta \cos \alpha$	(M1 A1)		
	Take moments about C:	$R_A\, a \cos \theta + W(a/2) \cos \theta$ $= R_B\, 2a \sin \theta$	(M1 A1)		
	Take moments about D:	$R_A\, 3a \cos \theta - W(3a/2) \cos \theta$ $= R_B\, 3a \sin \theta$	(M1 A1)		
	Solve for T, R_A, R_B (AEF in W and α):	$T = W / 2 \sin \alpha$ *or* $½W \operatorname{cosec} \alpha$ $R_A = 3W / 2$ $R_B = W / 2 \tan \alpha$ *or* $½W \cot \alpha$	B1 B1 B1	9	**9**
2	For A & B use conservation of momentum,	e.g.: $2mv_A + mv_B = 2mu$ (allow $2v_A + v_B = 2u$)	M1		
	Use Newton's law of restitution (consistent signs):	$v_B - v_A = eu$	M1		
	Combine to find v_A and v_B:	$v_A = (2 - e)\, u/3$, $v_B = 2(1 + e)\, u/3$	A1, A1	4	
	Find e from $v_A = \lvert v_B' \rvert$ with $v_B' =$ [–] $0{\cdot}4\, v_B$:	$(2 - e) = 0{\cdot}8\,(1 + e)$, $e = 2/3$	M1 A1	2	
	EITHER: Equate times in terms of reqd. distance x: [speeds need not be found: Use $v_A = v_B' = 0{\cdot}4\, v_B$ to solve for x:	$(d - x)/ v_A = d/v_B + x/v_B'$ (AEF) $v_A = v_B' = 4u/9$, $v_B = 10u/9$] $d - x = 0{\cdot}4\, d + x$, $x = 0{\cdot}3\, d$	M1 A1 M1 A1		
	OR: Find dist. moved by A when B reaches wall: Find reqd. distance x:	 $d_A = (d/v_B)\, v_A = 0{\cdot}4\, d$ $x = ½\,(d - d_A) = 0{\cdot}3\, d$	 (M1 A1) (M1 A1)	4	**10**

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3	Find k by equating equilibrium tensions: (vertical motion can earn M1 only)	$mg\,(a/2)/a = 2mg\,(3a/2 - ka)\,/\,ka$ $½ = 3/k - 2, \quad k = 6/5$ *or* $1{\cdot}2$	M1 A1 A1	3	
	Apply Newton's law at general point, e.g.: (lose A1 for each incorrect term)	$m\,\mathrm{d}^2x/\mathrm{d}t^2 = -mg\,(a/2 + x)/a$ $+ 2mg\,(3a/2 - ka - x)\,/\,ka$ *or* $m\,\mathrm{d}^2y/\mathrm{d}t^2 = +mg\,(a/2 - y)/a$ $- 2mg\,(3a/2 - ka + y)\,/\,ka$	M1 A2		
	Simplify to give standard SHM eqn, e.g.: **S.R.**: B1 if no derivation (max 2/5)	$\mathrm{d}^2x/\mathrm{d}t^2 = -(1 + 2/k)gx\,/\,a$ $= -8gx/3a$	A1		
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$: ($\sqrt{}$ on ω)	$T = 2\pi\sqrt{(3a/8g)}$ *or* $\pi\sqrt{(3a/2g)}$ *or* $3{\cdot}85\sqrt{(a/g)}$ *or* $1{\cdot}22\sqrt{a}$ [s]	B1$\sqrt{}$	5	
	Substitute values in $v^2 = \omega^2\,(x_0^{\,2} - x^2)$:	$0{\cdot}7^2 = (8g/3a)\{(0{\cdot}2a)^2 - (0{\cdot}05a)^2\}$	M1 A1		
	Solve to find numerical value of a:	$0{\cdot}49 = (8g/3) \times 0{\cdot}0375a, \; a = 0{\cdot}49$	A1	3	**11**

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4	*EITHER:* Find tension at top from $F = ma$ vertically:	$T = mu^2/a - mg$	B1		
	OR: Use energy at e.g. θ to upward vertical:	$\tfrac{1}{2} mv^2 = \tfrac{1}{2} mu^2 + mga(1 - \cos\theta)$			
	Find tension T by using $F = ma$ radially:	$T' = mv^2/a - mg\cos\theta$			
	Eliminate v^2:	$= mu^2/a + mg(2 - 3\cos\theta)$			
	Find T at top by taking $\theta = 0$:	$T = mu^2/a - mg$	(B1)		
	Find u_{min} by requiring $T \geqslant 0$ at top [or $T > 0$]:	$u^2/a - g \geqslant 0$ so $u_{min} = \sqrt{ag}$ **A.G.**	B1	2	
	Find v at bottom from conservation of energy:	$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 + mg \times 2a$	M1		
		$v^2 = ag + 4ag$, $v = \sqrt{(5ag)}$	A1		
	Find new speed V from conservation of momentum:	$m'V = mv$ with $m' = m + \tfrac{1}{4}m$	M1		
		$V = 4v/5 = 4\sqrt{(ag/5)}$ *or* $(4/5)\sqrt{(5ag)}$ AEF	A1	4	
	Find w^2 at angle θ from conservation of energy: (condone m instead of m' here since cancels out)	$\tfrac{1}{2} m'w^2 = \tfrac{1}{2} m'V^2 - m'ga(1 + \cos\theta)$	M1 A1		
		$[\, w^2 = ag(6/5 - 2\cos\theta)]$			
	S.R. Invalid energy method (max 2/5): [gives $T' = (5mg/4)(2 - 3\cos\theta)$]	$\tfrac{1}{2} m'w^2 = \tfrac{1}{2} mu^2 + mga(1 - \cos\theta) - \tfrac{1}{4}mga(1 + \cos\theta)$	(B1)		
	Find tension T' there by using $F = ma$ radially:	$T' = m'w^2/a - m'g\cos\theta$	B1		
	Eliminate w^2:	$= m'V^2/a - m'g(2 + 3\cos\theta)$	A1		
	Substitute for m' and V: AEF	$= (5mg/4)(6/5 - 3\cos\theta)$ *or* $3mg/2 - (15/4)mg\cos\theta$	A1	5	
	Find $\cos\theta$ when string becomes slack from $T' = 0$: **S.R.** Allow if found from $T' = mg(6/5 - 3\cos\theta)$	$\cos\theta = \tfrac{1}{3} \times 6/5 = 2/5$ *or* $0{\cdot}4$	M1 A1	2	**13**
5	Find or use sample mean	$\bar{x} = 222{\cdot}8 / 10 = 22{\cdot}28$			
	<u>and</u> estimate population variance: (allow biased here: $0{\cdot}412$ *or* $0{\cdot}642^2$)	$s^2 = 4{\cdot}12 / 9 = 0{\cdot}458$ *or* $103/225$ *or* $0{\cdot}677^2$	M1		
	Find confidence interval (allow z in place of t) e.g.:	$22{\cdot}28 \pm t\sqrt{(0{\cdot}458 / 10)}$	M1 A1		
	Use of correct tabular value:	$t_{9,\,0.975} = 2{\cdot}26[2]$	A1		
	Evaluate C.I. correct to 3 s.f.:	$22{\cdot}3 \pm 0{\cdot}48[4]$ *or* $[21{\cdot}8, 22{\cdot}8]$	A1	5	**5**

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6	Find prob. p of head from mean = 2 × variance:	$1/p = 2 \times (1-p)/p^2$, $p = ⅔$ **A.G.**	**M1 A1**	2	
(i)	Find $P(X = 4)$ (denoting $1 - p$ by q [= ⅓]):	$P(X = 4) = q^3 \times p$ $= 2/81$ *or* 0·0247	**B1**	1	
(ii)	Find or state $P(X > 4)$:	$P(X > 4)$ [$= 1 - (1 + q + q^2 + q^3) \times p$ $= 1 - (1 - q^4)$] $= q^4$ $= 1/81$ *or* 0·0123	**M1 A1**	2	
(iii)	Formulate condition for N: Take logs (any base) to give bound for N: Find N_{min}: ($N < 6{\cdot}29$ *or* $N = 6{\cdot}29$ earns M2 A0)	$1 - q^N > 0{\cdot}999$, [$(⅓)^N < 0{\cdot}001$] $N >$ log 0·001 / log ⅓ $N > 6{\cdot}29$, $N_{min} = 7$	**M1** **M1** **A1**	3	8
7	Find $F(x)$ for $1 \leqslant x \leqslant 4$:	$F(x) = (x^3 - 1)/63$	**B1**		
	Find $G(y)$ from $Y = X^2$ for $1 \leqslant x \leqslant 4$: (result may be stated)	$G(y) = P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})$ $= (y^{3/2} - 1)/63$	**M1 A1**		
	Find $g(y)$ for corresponding range of y:	$g(y) = y^{1/2}/42$ **A.G.**	**A1**		
	Find or state corresponding range of y :	$1 \leqslant y \leqslant 16$ **A.G.**	**B1**	5	
(i)	Find median value m of Y:	$(m^{3/2} - 1)/63 = ½$ $m = 32{\cdot}5^{2/3} = 10{\cdot}2$	**M1 A1**	2	
(ii)	Find $E(Y)$ [or equivalently $E(X^2)$]:	$E(Y) = \int y\, g(y)\, dy = \int y^{3/2}\, dy /42$ $= [y^{5/2}]_1^{16}/105 = 1023/105$ $= 341/35$ *or* 9·74	**M1 A1**	2	9

Question Number	Mark Scheme Details			Part Mark	Total
8	Find mean of sample data [for use in Poisson distn.]:	$\lambda = 220/100 = 2{\cdot}2$	**B1**		
	State (at least) null hypothesis (AEF):	H_0: Poisson distn. fits data *or* $\lambda = 2{\cdot}2$	**B1**		
	Find expected values $100\lambda^r e^{-\lambda}/r!$ (to 1 d.p.): (ignore incorrect final value here for M1)	11·080 24·377 26·814 19·664 10·8151 4·759 2·491	**M1 A1**		
	Combine last two cells so that exp. value ⩾ 5:	O_i: 3 E_i: 7·25	**M1***		
	Calculate value of χ^2 (to 2 d.p.; A1 dep M1*): (allow 7·95 if 1 d.p. exp.values used)	$\chi^2 = 0{\cdot}076 + 2{\cdot}879 + 0{\cdot}653 + 1{\cdot}448 + 0{\cdot}441 + 2{\cdot}491 = 7{\cdot}99$	**M1 A1**		
	State or use consistent tabular value (to 3 s.f.):	5 cells: $\chi_{3,0.95}{}^2 = 7{\cdot}815$ 6 cells: $\chi_{4,0.95}{}^2 = 9{\cdot}488$ (correct) 7 cells: $\chi_{5,0.95}{}^2 = 11{\cdot}07$	**B1**		
	State or imply valid method for conclusion e.g.:	Accept H_0 if χ^2 < tabular value	**M1**		
	Conclusion (AEF, requires both values correct): Not combining cells [so $\chi^2 = 8{\cdot}64$] can earn B1 B1 M1 A1 M0 M1 B1 M1 (max 7)	Distn fits *or* $\lambda = 2{\cdot}2$	**A1**	10	**10**
9	Calculate gradient b_1 in $y - \bar{y} = b_1(x - \bar{x})$:	$S_{xy} = 24\ 879 - 472 \times 400/8 = 1\ 279$ $S_{xx} = 29\ 950 - 472^2/8 = 2\ 102$ $b_1 = S_{xy} / S_{xx} = 0{\cdot}6085$ (3 s.f.)	**M1 A1**		
	Find regression line of y on x:	$y = 400/8 + b_1 (x - 472/8) = 50 + 0{\cdot}6085 (x - 59) = 0{\cdot}6085x + 14{\cdot}1$	**M1 A1**		
	Find y when $x = 72$:	$= 57{\cdot}9$ *or* 58			
	Allow use of regression line of x on y (since neither variable clearly independent):	$S_{yy} = 21\ 226 - 400^2/8 = 1\ 226$			
		$b_2 = S_{xy} / S_{yy} = 1{\cdot}043$	**(M1 A1)**		
		$x = 472/8 + b_2 (y - 400/8) = 1{\cdot}043\,y + 6{\cdot}85$	**(M1 A1)**		
		$Y = 62{\cdot}5$ *or* 62	**(A1)**	5	

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	Find product moment correlation coefficient r:	$r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}$ *or* $\sqrt{(0{\cdot}6085 \times 1{\cdot}043)} = 0{\cdot}797$	**M1 A1***		
	State both hypotheses (B0 for r …):	H_0: $\rho = 0$, H_1: $\rho \neq 0$	**B1**		
	State or use correct tabular two-tail r-value:	$r_{8,\,5\%} = 0{\cdot}707$	**B1***		
	State or imply valid method for conclusion e.g.:	Reject H_0 if $\lvert r\rvert$ > tab. value (AEF)	**M1**		
	Correct conclusion (AEF, dep A1*, B1*):	There is non-zero correlation	**A1**	6	**11**
10A	Find MI of lamina about Q:	$I_{\text{lamina}} = \frac{1}{3}m\{(3a)^2 + (3a/2)^2\} + m(9a/2)^2$ $[= (15/4 + 81/4)\,ma^2 = 24\,ma^2]$	**M1 A1**		
	State or find MI of rod about Q:	$I_{\text{rod}} = (\frac{1}{3} + 1)\,M\,(3a/2)^2$ $[= 3Ma^2]$	**B1**		
	Sum to find MI of object about Q:	$I_1 = 24\,ma^2 + 3\,Ma^2$ $= 3\,(8m + M)\,a^2$ **A.G.**	**A1**	4	
	Find MI of object about mid-point of PQ:	$I_2 = (15/4 + 3^2)\,ma^2 + \frac{1}{3}\,M\,(3a/2)^2$ $= (51/4)\,ma^2 + \frac{3}{4}\,Ma^2$ $= \frac{3}{4}\,(17m + M)\,a^2$ **A.G.**	**M1 A1**	2	
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_1:	$[-]I_1\,d^2\theta/dt^2 = mg \times (9a/2)\sin\theta + Mg \times (3a/2)\sin\theta$ $[= (9m/2 + 3M/2)\,ga\sin\theta]$	**M1 A1**		
	[Approximate $\sin\theta$ by θ and] find $\omega_1^{\,2}$ in SHM eqn:	$\omega_1^{\,2} = (3m + M)g / 2(8m + M)\,a$	**M1**		
	Find period T_1 for axis l_1 from $2\pi/\omega_1$: (AEF)	$T_1 = 2\pi\sqrt{\{2(8m + M)\,a / (3m + M)g\}}$	**A1**		
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_2:	$[-]I_2\,d^2\theta/dt^2 = mg \times 3a\sin\theta$	**M1**		
	[Approximate $\sin\theta$ by θ and] find $\omega_2^{\,2}$ in SHM eqn:	$\omega_2^{\,2} = 4mg / (17m + M)\,a$	**M1**		
	Find period T_2 for axis l_2 from $2\pi/\omega_2$: (AEF)	$T_2 = 2\pi\sqrt{\{(17m + M)\,a / 4mg\}}$	**A1**		
	Verify that $T_1 = T_2$ when $m = M$: (AEF) [Taking $m = M$ throughout 2nd part can earn: M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)]	$T_1 = 2\pi\sqrt{(18\,a / 4g)} = T_2$	**B1**	8	**14**

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10B	State hypotheses (B0 for $\bar{x}$ …), e.g.:	H_0: $\mu_X = \mu_Y$, H_1: $\mu_X \neq \mu_Y$	**B1**		
	State assumption (AEF):	Distributions have equal variances	**B1**		
	Find sample means <u>and</u> estimate popln. variances: (allow biased here: 0·36 *or* $0{\cdot}6^2$) (allow biased here: 0·4183 *or* $0{\cdot}6468^2$)	$\bar{x} = 4{\cdot}2$, $\bar{y} = 4{\cdot}8$ $s_X^2 = (180 - 42^2/10) / 9$ $= 0{\cdot}4$ *or* $0{\cdot}6325^2$ $s_Y^2 = (281{\cdot}5 - 57{\cdot}6^2/12) / 11$ $= 0{\cdot}4564$ *or* 251/550 *or* $0{\cdot}6755^2$	**M1**		
	Estimate (pooled) common variance: (note s_X^2 and s_Y^2 not needed explicitly)	$s^2 = (9\, s_X^2 + 11\, s_Y^2) / 20$ (AEF) *or* $(180 - 42^2/10 + 281{\cdot}5 - 57{\cdot}6^2/12) / 20$ $= 0{\cdot}431$ *or* $0{\cdot}6565^2$	**M1 A1**		
	Calculate value of t (to 3 s.f.):	[-] $t = (\bar{y} - \bar{x}) / s\,\sqrt{(1/10 + 1/12)}$ $= 2{\cdot}13$	**M1 A1**		
	State or use correct tabular t value: (or can compare $\bar{y} - \bar{x} = 0{\cdot}6$ with 0·586)	$t_{20,\, 0.975} = 2{\cdot}086$ [allow 2·09]	**B1***		
	Correct conclusion (AEF, √ on t, dep *B1): **S.R.** Implicitly taking s_X^2, s_Y^2 as popln. variances: (may also earn first B1 B1 M1) Comparison with $z_{0.975}$ and conclusion: (can earn at most 5/9)	$t > 2{\cdot}09$ so mean masses not same $z = (\bar{y} - \bar{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)}$ $= 0{\cdot}6 / \sqrt{(0{\cdot}078} = 2{\cdot}15$ $2{\cdot}15 > 1{\cdot}96$ so mean masses not same	**B1√** **(B1)**	9	
	State hypotheses (B0 for $\bar{x}$ …), e.g.:	H_0: $\mu_X = 3{\cdot}8$, H_1: $\mu_X > 3{\cdot}8$ *or* H_0: $\mu_X = \mu_Z$, H_1: $\mu_X > \mu_Z$	**B1**		
	Calculate value of t using s_X from above:	$t = (4{\cdot}2 - 3{\cdot}8) / (s_X/\sqrt{10}) = 2{\cdot}0$	**M1 A1**		
	State or use correct tabular t value: (or can compare 0·4 with 0·367)	$t_{9,\, 0.95} = 1{\cdot}833$ [allow 1·83]	**B1***		
	Correct conclusion (A.E.F., √ on t, dep *B1):	$t > 1{\cdot}833$, so claim is justified *or* mean mass of Royals > mean mass of Crowns	**B1√**	5	**14**