FURTHER MATHEMATICS

Paper 9231/11

Paper 11

Key messages

- A considerable proportion of the total marks in this paper depend on giving answers which are correct to 3 significant figures. In order to gain as many marks as possible all working should be either exact or correct to at least 4 significant figures.
- Candidates need to be careful to not spend too much time answering any one question.

General Comments

The scripts for this paper were generally of good quality. There were a number of high quality scripts and many candidates showed evidence of sound learning. Working was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a sound standard. Work on calculus and matrices was of a high standard.

A very high proportion of scripts had substantial attempts at all eleven questions. There were few misreadings of the questions and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the calculus and matrices work, already mentioned, the question on rational functions was done well, as was the proof by induction question. Candidates need to make sure they fully understand polar coordinates.

Comments on specific questions

Question 1

There were variable responses to this question. The best candidates wrote down equations for $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\alpha\beta\gamma$, remembering that $\gamma = \alpha$ (say), because it was given that the equation had a repeated root. Then, by eliminating α and β, they obtained the relationship between p and q. Some candidates expressed the cubic equation as $f(x) = 0$ and used the fact that the repeated root was a solution of $f(x) = 0$ and $f'(x) = 0$. Some candidates chose to express $4p^3 + 27q^2$ in terms of α and β. This gave a sixth order expression. If they chose α as their repeated root, it had factors of α^3 and $2\alpha + \beta$, the latter implying that the expression was zero, from the sum of the roots. Weaker candidates did not use the 'repeated root' information correctly, either by continuing with three distinct roots, or by having three equal roots. A small minority failed to make any progress with the question.

Question 2

Almost all candidates found $\mathbf{a} \times \mathbf{b}$ correctly. Some candidates did not realise that half the magnitude of this vector gave the area of triangle OAB. Either they forgot the half factor, giving an incorrect value, or they used an altogether different method. Attempts to find the perpendicular height of the tetrahedron $OABC$ were often incorrect. Few realised that they required $|\mathbf{c} \,.\, \hat{\mathbf{n}}|$, where $\mathbf{n} = \mathbf{a} \times \mathbf{b}$.

Answers: $\mathbf{i} - 10\mathbf{j} - 17\mathbf{k}$, $\frac{1}{2}\sqrt{390}$ or 9.87, 5.

Question 3

There were good attempts at proof by induction in this paper. Most candidates stated a correct inductive hypothesis, H_k, and were able to prove H_1 was correct. Many realised that the expression assumed true in H_k had to be differentiated once more. Only the better candidates were able to complete the proof, some losing the final mark for an inadequate conclusion, which required, at least, a mention that the result was true for all positive integers.

Question 4

This question was mostly well answered. The matrix **M** was invariably reduced to a correct echelon form, from which its rank was deduced and a suitable basis for the range space of the linear transformation T was stated. Candidates then wrote down two linear equations from which they mostly deduced a correct basis for the null space of T. There were very few errors with arithmetic or algebra.

Answers: **(i)** 2, $\left\{\begin{pmatrix}3\\6\\9\\15\end{pmatrix},\begin{pmatrix}4\\7\\9\\16\end{pmatrix}\right\}$; **(ii)** $\left\{\begin{pmatrix}1\\-2\\0\\1\end{pmatrix},\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}-3\\0\\2\\1\end{pmatrix},\begin{pmatrix}0\\-3\\1\\2\end{pmatrix}\right\}$

or any two of the above four vectors

Question 5

There was some extremely accurate work in responses to this question. There were only rare instances of incorrect work in the first part of the question. In the second part, the differentiation of product terms occasionally produced an error. A small number of candidates lost only the final mark due to a small error with their arithmetic.

Answers: **(i)** $\frac{3}{4}$; **(ii)** $\frac{1}{16}$

Question 6

There were many completely correct answers to this question. Marks were most commonly lost in deducing the reduction formula, where, having made a correct start by integrating by parts, candidates did not see how to re-write $(x-1)^{\frac{3}{2}}$ as $(1 - x)\ (x-1)^{\frac{1}{2}}$ and rearrange the equation. In the final part, a small minority made a sign error.

Answer: $\frac{32}{315}$

Question 7

The equations of the asymptotes were invariably found correctly. In a few instances a slip occurred with the oblique asymptote. In the middle part of the question, most candidates attempted to find a quadratic equation by using the fact that $\frac{dy}{dx}=0$ for turning points. Usually they proceeded to say that $B^2-4AC>0$ for distinct roots. The use of = or ≥ was incorrect. The sketch graph often lacked precision, particularly in the forms at infinity. Some candidates did not adequately demonstrate that there was no intersection of the graph with the x-axis, although they usually found the y-intercept correctly. The equation of the oblique asymptote was sometimes shown as $y = x - 1$ rather than $y = x + 1$ in the case $p = -1$.

Answers: $x = 2$, $y = x + p + 2$; $p > -\frac{5}{2}$; (0,–0.5)

Question 8

Only the better candidates were able to produce a satisfactory proof of the general result. Weaker candidates made little progress with this proof, while others made false assumptions about commutativity. Almost all candidates were able to identify the eigenvalues of the matrix **C**. They then usually made sound attempts to find the eigenvectors. Disappointingly, a significant minority did not appreciate the need to pre-multiply the given eigenvector by matrix **D**, but instead chose to find all the eigenvalues and eigenvectors of matrix **D**. In this case the marks were earned, but time was wasted. Some of these candidates also repeated the exercise for the matrix **CD**. In this case marks were not awarded, as the 'hence' instruction was being ignored. There were, however, many candidates who found eigenvalues and eigenvectors correctly.

Answers: –1, 1, 2; $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-2\\0\end{pmatrix} \begin{pmatrix}1\\6\\3\end{pmatrix}$; –2; $\begin{pmatrix}1\\6\\3\end{pmatrix}$, –4

Question 9

As with other calculus questions on the paper, there were many complete and accurate answers to this question. Those difficulties emerging were largely due to not being able to find the square root of $1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$. This, in turn, meant that little progress could be made with integration in parts **(ii)** and **(iii)** of the question.

Answers: **(i)** $\frac{12}{2\ln 5}\ (=1.49)$; **(ii)** $\frac{12}{5}$; **(iii)** $\pi\left(\frac{156}{25}+\ln 5\right)\ (=24.7)$.

Question 10

The sketch graphs produced for this question were frequently inaccurate. Scales were often missing. The curve C often did not pass through the point $(1, \pi)$. If it did so, the shape of the graph was frequently incorrect at this point. There was sometimes an extra loop for C, corresponding to negative r values. The convention for this syllabus is $r > 0$. Despite the question saying that l was a straight line, sometimes it was drawn as a curve. It often had the wrong position. When stating the polar coordinates of the points of intersection of C and l, one point was frequently omitted, or the r and θ values were given in the wrong order. In the final part of the question, the region R was misidentified. Some credit was given for the correct use of the double angle formula in integrating $\frac{1}{2}r^2$ with respect to θ. There were many solutions with no mention of the triangular piece of the region R.

Answers: $\left(4, \frac{\pi}{3}\right), \left(4, -\frac{\pi}{3}\right)$; $\frac{22\pi}{3}-\frac{5\sqrt{3}}{2}$ (= 18.7).

Question 11

EITHER

This was the less popular question of the two alternatives. Those who did this alternative, in fact, did it well. They were able to use De Moivre's theorem in order to obtain the introductory results. Direct calculations enabled them to derive the next two results, which enabled them to find the sum and product of $\cos\frac{1}{5}\pi$ and $\cos\frac{3}{5}\pi$. Those who got to this point were able to construct the required quadratic equation and choose the correct root for the value of $\cos\frac{1}{5}\pi$.

Answers: $\frac{1}{2}, -\frac{1}{4}$; $x=\frac{2+2\sqrt{5}}{8}$; $\frac{1+\sqrt{5}}{4}$.

OR

Almost all candidates were able to find $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$. Some candidates did not then show sufficient working to obtain the given *z*-*x* differential equation. Only a small minority found $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of *z* and *x*, which they then substituted into the given equation. This approach is more complicated and runs the risk of making an error. Many candidates found the general solution of the *z*-*x* differential equation correctly. There were a few candidates with the wrong form for the complementary function or the particular integral. The latter was more common than the former. Usually the particular integral was expressed as kx^2 only. Few candidates did not express *y* in terms of *x*. Marks were lost in the final part of the question by those candidates who did not indicate which terms in the solution were tending to zero and thence what was the resulting limit for *y*.

Answers: $y = \frac{A}{x^2}e^{-x} + \frac{B}{x}e^{-2x} + 2 - \frac{4}{x} + \frac{4}{x^2}$;

As $x \to \infty$, $e^{-2x}, \frac{1}{x}$ and $\frac{1}{x^2} \to 0 \therefore y \to 2$

FURTHER MATHEMATICS

Paper 9231/12

Paper 12

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As $x \to \infty$, $e^{-2x}, \frac{1}{x}$ and $\frac{1}{x^2} \to 0 \therefore y \to 2$

FURTHER MATHEMATICS

Paper 9231/13

Paper 13

Key messages

- Candidates need to be careful to not spend too much time answering any one question.
- Candidates need to be careful to eliminate arithmetic and numerical errors in their answers.

General Comments

There was a high standard of work in answers to this paper, and some candidates made very good responses to the questions. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a high standard. Calculus work, in particular, was very impressive.

All of the scripts had substantial attempts at all eleven questions. There were no misreadings and no rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the calculus work, already mentioned, candidates tackled the questions on series, roots of equations and vectors confidently. This group of candidates were able to cope well with the questions on complex numbers, proof by induction and polar curves, which are often a challenge for candidates.

Comments on specific questions

Question 1

The initial result was easily verified by these candidates. They then used it to express the series as a sum of differences and were able to find the sum of the series in terms of N. The final mark was sometimes lost when, having established correctly, that $(N+1) > 10^8$, the least N was not given as 10^8.

Answers: $S_N = 1 - \frac{1}{(N+1)^2}$, 10^8

Question 2

The proof by induction was well done and candidates realised that to prove that $H_k \Rightarrow H_{k+1}$ involved differentiating the result assumed true in the inductive hypothesis once more. The final mark was sometimes lost for the omission, or incompleteness, of the conclusion.

Question 3

All candidates were able to obtain, correctly, the sum of the squares of the roots of the cubic equation. They were also able to expand the determinant of the given matrix. The majority were able to complete the question, but a small number did not see how to show that the determinant was zero, using the result from the first part of the question.

Answer: 31

Question 4

The first part of the question was invariably done correctly. Mostly, the second part was also done correctly, with the most common error being the sign of $\sec^3 2t$, which resulted in the final answer having the wrong sign. Only the occasional candidate demonstrated lack of familiarity with these trigonometric functions. Working with cartesian equations was occasionally seen. The mark scheme has an alternative solution for this case.

Answer: **(i)** $\frac{3\sqrt{3}}{2}$; **(ii)** 6.

Question 5

The majority of candidates knew that if $z = \cos\theta + \mathrm{i}\sin\theta$ then $z + \frac{1}{z} = 2\cos\theta$ and were able to use it, correctly, to obtain $\cos^4\theta$. Those who did not know this result had sufficiently good algebra to get the required result by longer methods, or very near to it. Candidates were then able to use their result to evaluate the integral in the correct manner.

Answers: $a = \frac{1}{8}, b = \frac{1}{2}, c = \frac{3}{8}$, $\frac{1}{4} + \frac{3\pi}{32}$

Question 6

It was not unusual for the answer to this question to be completely correct. A small number of candidates were not able to describe the behaviour of x as $t \to \infty$.

Answers: $x = A\mathrm{e}^{-2t} + Bt\mathrm{e}^{-2t} - \frac{1}{8}\cos 2t$, x oscillates or $x \to -\frac{1}{8}\cos 2t$

Question 7

Most candidates differentiated the given expression, but only a few were able to see the rearrangement which gave the printed result. Whether this was obtained, or not, candidates were then able to use the printed result in order to obtain the reduction formula. Most candidates were able to find either I_0 or I_1 correctly and use the reduction formula to find I_3, as required.

Answer: $\frac{319}{140}$ or 2.28

Question 8

Most candidates were able to correctly sketch the graph. Graphs need to conform to the stated range for θ. The integration of $\frac{1}{2}\int r^2\,\mathrm{d}\theta$ was mostly correct, with the appropriate double angle formula being used. A few candidates managed to get the wrong limits for either, or both, of A_1 or A_2, thus leading to the incorrect final ratio.

Answer: 12.2

Question 9

Candidates gave good answers to this question. The cartesian equation of the plane Π containing the two given lines was invariably found correctly. The likely source of any error was a wrong sign or constant term. The majority knew how to find the vector equation of the line l, and substitute its components into the cartesian equation of the plane Π, thus enabling them to use the value of the parameter in order to find the coordinates of the intersection of l and Π. The stronger candidates used either the triple scalar product, or the distance of a point from a plane formula in order to find the perpendicular distance of the point P from Π. Others used a method involving the use of Pythagoras in a relevant right-angled triangle. This was often done correctly, but used up more time. In the final part, most used the scalar product to find the complement of the angle between l and Π. A few candidates forgot to subtract the angle from 90º, or exchange $\sin\theta$ for $\cos\theta$ in their scalar product formula.

Answers: **(i)** $4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$; **(ii)** $\frac{21}{\sqrt{131}}$ or 1.83; **(iii)** 23.6º

Question 10

Finding the coordinates of the intersection of C with the axes posed no difficulty for this group of candidates. Most used a method involving the discriminant of a quadratic equation to show $-1 \le y \le 15$. Those who merely found the coordinates of the turning points needed to refer to the graph, or say something about having a continuous function, in order to gain the final mark. The graph was generally correct, the equation of the horizontal asymptote, however, was sometimes incorrect.

Answers: (–1,0), (2,0), (0, –1); (–4,15), (0, –1), $y = 5$.

Question 11

EITHER

This was the more popular alternative for this final question. Most candidates knew the methods required in all sections of the question. The errors that occurred were mainly small slips with arithmetic, or algebra. The weaker candidates had difficulty in finding the square root of their expression for $\frac{ds}{dx}$ and so were unable to integrate their expressions for the arc length and surface area.

Answers: $\frac{4\sqrt{3}}{15}$ or 0.462, $2\sqrt{3}$ or 3.46, 3π or 9.42.

OR

Those attempting this alternative were able to find the eigenvalues and eigenvectors of the matrix **A** accurately. The middle section of the question caused some uncertainty. There was a mark for stating the equation of the plane Π in the form $\mathbf{r} = s\mathbf{e} + t\mathbf{f}$ and two further marks for showing, with appropriate working, that $\mathbf{Ar} = (s\lambda)\mathbf{e} + (t\mu)\mathbf{f}$. Whether the middle section was accomplished, or not, good attempts were made to obtain the equations of the three planes in the final section of the question, usually by finding their normal vector from the vector product of two of the eigenvectors.

Answers: 1, 2, 3; $-2\mathbf{j} + \mathbf{k}$, $\mathbf{i} + \mathbf{j}$, $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$; $x - y - 2z = 0$, $2x - y - 2z = 0$, $x - y = 0$

FURTHER MATHEMATICS

Paper 9231/21

Paper 21

Key Messages

Since only one question on the paper offers a choice, the two topics of Mechanics and Statistics each comprise at least 43% of the available credit. Candidates wishing to perform well overall must be well prepared in both topics.

General comments

Many candidates showed evidence of having mastered all parts of the syllabus, and consequently produced a good overall performance. While the Statistics option was markedly more popular in the final question, this most likely reflected the apparent relative difficulty of the two alternatives rather than any general weakness in Mechanics. Candidates' work was often well presented.

Comments on specific questions

Question 1

Although the particle is moving in a circle, candidates should realise that the reference to its acceleration in the question is to its linear acceleration at the specified time, rather than the angular acceleration of a notional radius. This requires combining the transverse component $2t$ and the radial component $(t^2 - 12)^2/2$ when $t = 4$, rather than finding only one or the other of these two components.

Answer: $8\sqrt{2}$ m s^{-2}

Question 2

The majority of candidates applied the standard Simple Harmonic Motion (SHM) formula $v^2 = \omega^2(x_0^{\,2} - x^2)$ correctly for each of the two specified values 5 and 9 of the displacement x. Both V and ω may be eliminated from the resulting simultaneous equations, verifying the given amplitude x_0. Taking v to be the given greatest speed $3\sqrt{2}$ when $x = 0$, yields a value 2/5 for ω, enabling V to be found from either of the simultaneous equations.

Answer: $\sqrt{14}$

Question 3

The first part of this question was often answered correctly by using the conservation of energy to effectively express v^2 as $2ag\ (1 - \cos\theta)$, and by equating radial forces to give $v^2 = ag\cos\theta$ since the reaction is zero when contact is lost. The given value ⅔ of $\cos\theta$ then results immediately. The simplest way to find the greatest height reached by the particle is to first find the vertical component of the particle's speed when contact is first lost, namely $v\sin\theta$, and hence the further height $5a/27$ risen using $(v\sin\theta)^2/2g$. A second application of conservation of energy will not succeed if the greatest height reached is wrongly assumed to be that at which the kinetic energy is zero, corresponding to zero speed, since the particle is in free flight rather than constrained to continue moving along the surface of the sphere. The final step is to include the height $a\cos\theta$ above the level of O of the point at which contact is lost.

Answer: **(ii)** $23a/27$

Question 4

The given expression for the coefficient of restitution was verified by almost all candidates using conservation of momentum and Newton's law of restitution. In the second part of the question it is essential to understand that the total kinetic energy of the particles consists only of the kinetic energies before the collision and not the sum of the kinetic energies before and after the collision. The simplest approach is to equate the kinetic energy of Q after the collision to two-thirds of the sum of P and Q's kinetic energies prior to the collision, though occasionally candidates misplaced the two-thirds factor. Writing down the kinetic energy lost and equating it one third of the initial energy is of course equally acceptable, but slightly increases the risk of error in the subsequent working. Finally replacing Q's speed after the collision by $(\frac{1}{3} + k)u$ produces the required given value of k.

Question 5

The moment of inertia of the sphere about its centre C may be found using the appropriate expression in the *List of Formulae*, and an application of the parallel axes theorem then yields its moment of inertia about the specified axis through O. That for the system follows, by adding P's moment of inertia $9Ma^2$. The key to the remainder of the question is to formulate the equation of angular motion for the system, and approximate $\sin \theta$ by θ in order to produce the standard SHM equation $d^2\theta/dt^2 = -\omega^2\theta$ with here $\omega^2 = 5g/14a$. The period can then be found from $2\pi/\omega$, and the time from $(1/\omega) \cos^{-1} (\frac{1}{2})$. Candidates should realise that if instead conservation of energy is used to produce an equation for the angular velocity $d\theta/dt$, which they choose to denote by ω, it is an entirely different entity to the ω conventionally used in the standard form of the SHM equation and cannot therefore be used as such. Indeed the standard SHM results cannot be applied since the motion has not been shown to be simple harmonic by this approach.

Answers: **(i)** $2\pi \sqrt{(14a/5g)}$; **(ii)** $\frac{1}{3} \pi \sqrt{(14a/5g)}$

Question 6

As well as integrating $f(x)$ for $1 \le x \le 3$ and determining the constant of integration from $F(0) = 0$ or $F(3) = 1$, candidates should state the value of $F(x)$ for the other two intervals of x. Most candidates were able to find the distribution function of Y correctly and differentiate it to find the probability density function $g(y)$, but for completeness the relevant interval of validity should be stated in terms of y. The expected value $E(Y)$ of Y is found in the usual way by integrating $yg(y)$ over (1, 27), and a similar integration of $y^2g(y)$ with the sometimes forgotten subtraction from it of the square of $E(Y)$ gives the required variance. Alternatively the expected value and variance of X^3 can be found.

Answers: **(i)** $1/6y^{-2/3}$ $(1 \le y \le 27)$, 0 otherwise ; **(ii)** 10, 393/7

Question 7

Most candidates realised immediately that λ must be the reciprocal of the specified mean lifetime, and many found the first required probability $p = e^{-0.5}$ by integrating $f(t)$ to give the distribution function $F(t)$, and then evaluating $1 - F(1000)$. The probability that no more than one of the bulbs has a lifetime less than 1000 hours appeared to be a little more challenging, however. The correct answer is found from $p^6 + 6p^5(1-p)$, but a variety of incorrect attempts were seen, some with p and $1-p$ interchanged. Candidates should find the least value of the new mean lifetime by formulating an inequality for the new λ using the given information, and then take the reciprocal of the resulting upper bound on λ. Working instead with an equality is only valid if accompanied by an adequate justification for accepting the resulting $1/\lambda = -4/\ln 0{\cdot}999$ as the least value of the new mean lifetime.

Answer: 0·607; 0·244; 4000 hours

Question 8

The great majority of candidates realised that they should integrate $f(x)$ over the whole or a part of the interval (0, 6) and equate the result to an appropriate given number in order to find the value of k. In the case of (0, 6), for example, the integral should be equated to the total number 216 of observations, while the integral over (0, 1) should be equated to the expected fraction 1/216 of the observations, giving in all cases $k = 1/72$. The expected frequencies a, b and c then follow by integrating 216 $f(x)$ over the appropriate intervals, or in the case of the last one found, subtracting all the others from the known total 216. A great many candidates, however, took $f(x)$ to be a formula for calculating the expected frequencies instead of the

probability density function for the distribution of X, and thus equated the integral over (0, 1) to the tabular value 1, for example. The values of a, b and c resulting from this invalid process happen to be in agreement with the correct ones and the goodness of fit test is unaffected. When conducting a χ^2 test, candidates should be prepared to combine adjacent cells if necessary in order to ensure that all the expected frequencies (but not necessarily the observed ones) are at least 5. In this case the test requires the combining of the first two cells, a statement of at least the null hypothesis, and then a comparison of the calculated value 6·69 of χ^2 with the tabular value 7·779. This leads to an acceptance of the null hypothesis, namely that f(x) fits the data.

Answer: **(i)** $b = 37$, $c = 61$; **(ii)** f(x) fits the data

Question 9

Most candidates understood how to find the required confidence interval, though instead of the correct tabular t-value 2·776 some used another t-value or even a z-value. The hypotheses in the test should be stated in terms of the means of the populations and not of the samples, and the test is of course here a two-sided one. In order to perform the test, candidates should estimate the unknown common population variance s^2 from the two samples, obtaining a value 13·05, rather than simply using the unbiased estimate 13·8 found from the 5-rod sample in the first part of the question. Comparison of $(524{\cdot}6 - 521{\cdot}6)/(s\sqrt{(5^{-1} + 10^{-1})}) = 1{\cdot}52$ with the tabular t-value 1·771 leads to the conclusion that there is no difference in the population means.

Answer: (520, 529)

Question 10

EITHER

The Mechanics option was much less popular than the Statistics one, and serious attempts at all parts were rare. As in all such questions, it is advisable for candidates to show the various forces on a diagram or at least to use easily recognisable symbols such as R_A for the normal reaction at A. It is possible that some candidates write this information on the diagram in the question paper, but they should realise that this is not seen in marking. It is also advisable to first consider what unknown forces need to be found in each part of the question, and then choose moment or resolution equations accordingly, rather than writing down and manipulating randomly chosen equations in the hope that the required answer will emerge. Thus in the first part of the question, taking moments about P and then B for the rod will yield the friction and normal reaction at A respectively, from which the required inequality for μ follows. Having noted that the tension T in the string must be kW, there are a variety of possible moment or resolution equations which give the value of k, of which the simplest is probably taking moments for the rod about A. The answer to the final part is clearly found from $T \sin \theta$.

Answer: **(i)** $\sqrt{(5/6)}$; **(ii)** $\sqrt{(5W/9)}$.

OR

The equation in part **(i)** was shown to be true by most candidates, usually by inserting the mean values of x and y into the regression line of y on x or, equivalently but much less frequently, utilising the first of the normal equations for a least squares approximation. Although many candidates realised that, equating 2·5 to the usual formula for the coefficient b_1 of the regression line (or equivalently using the second normal equation), would yield a second linear equation in p and q, and hence their values. Arithmetical errors were sometimes made in the process. Finding the value of the product moment correlation coefficient is a straightforward use of the corresponding equation in the *List of Formulae*, and usually presented little difficulty provided the correct values of p and q had been found. The equivalent but less direct approach of finding the coefficient b_2 of the regression line of x on y, and then evaluating $\sqrt{(b_1 b_2)}$ was sometimes seen. While many candidates stated correctly in the final part of the question that the value of the product moment correlation coefficient is unchanged, fewer were able to state correctly the equation of the regression line.

Answers: **(ii)** $p = 4$, $q = 15$; **(iii)** 0·953; **(iv) (a)** $y = 0{\cdot}25x - 1{\cdot}5$, **(b)** 0·953

FURTHER MATHEMATICS

Paper 9231/22

Paper 22

Key Messages

Since only one question on the paper offers a choice, the two topics of Mechanics and Statistics each comprise at least 43% of the available credit. Candidates wishing to perform well overall must be well prepared in both topics.

General comments

Many candidates showed evidence of having mastered all parts of the syllabus, and consequently produced a good overall performance. While the Statistics option was markedly more popular in the final question, this most likely reflected the apparent relative difficulty of the two alternatives rather than any general weakness in Mechanics. Candidates' work was often well presented.

Comments on specific questions

Question 1

Although the particle is moving in a circle, candidates should realise that the reference to its acceleration in the question is to its linear acceleration at the specified time, rather than the angular acceleration of a notional radius. This requires combining the transverse component $2t$ and the radial component $(t^2 - 12)^2/2$ when $t = 4$, rather than finding only one or the other of these two components.

Answer: $8\sqrt{2}\ \text{m s}^{-2}$

Question 2

The majority of candidates applied the standard Simple Harmonic Motion (SHM) formula $v^2 = \omega^2(x_0^{\,2} - x^2)$ correctly for each of the two specified values 5 and 9 of the displacement x. Both V and ω may be eliminated from the resulting simultaneous equations, verifying the given amplitude x_0. Taking v to be the given greatest speed $3\sqrt{2}$ when $x = 0$, yields a value 2/5 for ω, enabling V to be found from either of the simultaneous equations.

Answer: $\sqrt{14}$

Question 3

The first part of this question was often answered correctly by using the conservation of energy to effectively express v^2 as $2ag\,(1 - \cos\theta)$, and by equating radial forces to give $v^2 = ag\cos\theta$ since the reaction is zero when contact is lost. The given value ⅔ of $\cos\theta$ then results immediately. The simplest way to find the greatest height reached by the particle is to first find the vertical component of the particle's speed when contact is first lost, namely $v\sin\theta$, and hence the further height $5a/27$ risen using $(v\sin\theta)^2/\,2g$. A second application of conservation of energy will not succeed if the greatest height reached is wrongly assumed to be that at which the kinetic energy is zero, corresponding to zero speed, since the particle is in free flight rather than constrained to continue moving along the surface of the sphere. The final step is to include the height $a\cos\theta$ above the level of O of the point at which contact is lost.

Answer: **(ii)** $23a/27$

Question 4

The given expression for the coefficient of restitution was verified by almost all candidates using conservation of momentum and Newton's law of restitution. In the second part of the question it is essential to understand that the total kinetic energy of the particles consists only of the kinetic energies before the collision and not the sum of the kinetic energies before and after the collision. The simplest approach is to equate the kinetic energy of Q after the collision to two-thirds of the sum of P and Q's kinetic energies prior to the collision, though occasionally candidates misplaced the two-thirds factor. Writing down the kinetic energy lost and equating it one third of the initial energy is of course equally acceptable, but slightly increases the risk of error in the subsequent working. Finally replacing Q's speed after the collision by $(\frac{1}{3} + k)u$ produces the required given value of k.

Question 5

The moment of inertia of the sphere about its centre C may be found using the appropriate expression in the *List of Formulae*, and an application of the parallel axes theorem then yields its moment of inertia about the specified axis through O. That for the system follows, by adding P's moment of inertia $9Ma^2$. The key to the remainder of the question is to formulate the equation of angular motion for the system, and approximate $\sin\theta$ by θ in order to produce the standard SHM equation $d^2\theta/dt^2 = -\omega^2\theta$ with here $\omega^2 = 5g/14a$. The period can then be found from $2\pi/\omega$, and the time from $(1/\omega)\cos^{-1}(\frac{1}{2})$. Candidates should realise that if instead conservation of energy is used to produce an equation for the angular velocity $d\theta/dt$, which they choose to denote by ω, it is an entirely different entity to the ω conventionally used in the standard form of the SHM equation and cannot therefore be used as such. Indeed the standard SHM results cannot be applied since the motion has not been shown to be simple harmonic by this approach.

Answers: **(i)** $2\pi\sqrt{(14a/5g)}$; **(ii)** $\frac{1}{3}\pi\sqrt{(14a/5g)}$

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As well as integrating $f(x)$ for $1 \le x \le 3$ and determining the constant of integration from $F(0) = 0$ or $F(3) = 1$, candidates should state the value of $F(x)$ for the other two intervals of x. Most candidates were able to find the distribution function of Y correctly and differentiate it to find the probability density function $g(y)$, but for completeness the relevant interval of validity should be stated in terms of y. The expected value $E(Y)$ of Y is found in the usual way by integrating $yg(y)$ over (1, 27), and a similar integration of $y^2g(y)$ with the sometimes forgotten subtraction from it of the square of $E(Y)$ gives the required variance. Alternatively the expected value and variance of X^3 can be found.

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FURTHER MATHEMATICS

Paper 9231/23

Paper 23

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Answer: $8\sqrt{2}$ m s^{-2}

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