FURTHER MATHEMATICS

Paper 9231/11
Paper 11

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should be confident in all aspects of the curriculum, and be able to apply their understanding and knowledge to solving problems. This should include knowledge from the A Level Mathematics syllabus as well as the Further Mathematics syllabus.
- Candidates should learn any standard formulae that are not on the formula sheet.

General comments

Most candidates presented their work clearly and logically, but there were a number of scripts which did not show good mathematical communication. Gaps in knowledge were evident in some scripts.

Comments on specific questions

Question 1

(i) Most candidates formed a matrix with the 3 vectors and either found its determinant or reduced row echelon form, and some candidates clearly related this to the independence of the three vectors. Others included reference to the dimension of the underlying vector space. It was not necessary to include the 4th vector in this matrix.

(ii) Most candidates were able to find the relationship between the four vectors either by setting up and solving equations or by inspection.

Answers: **(ii)** $\mathbf{d} = -\mathbf{a} - \mathbf{b} + \mathbf{c}$

Question 2

(i) and (ii) Both parts of this question were very well done by the majority of candidates who correctly recalled and applied the relationships between the roots and coefficients of the cubic equation and were able to show that the properties given held.

Question 3

(i) The majority of candidates followed the method suggested and considered the expression for $3 - u_{k+1}$ and then used the given inequality. However a number of candidates did not complete all the steps of an induction proof and some tried to prove the given iterative formula rather than the required inequality. Care must be taken over the starting point – in this case it was given in the question.

(ii) Many candidates substituted in the iterative formula and recognised that $-u_n^2 + 9 < 0$.

Question 4

(i) Although most candidates tried to find $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$, some found it difficult to simplify the resulting expression, and others cancelled out the factor of 2 rather than retaining it in their final answer. Some candidates recalled the formula for area of surface correctly, but others omitted or misplaced the variable, *y*.

(ii) Most candidates were able to use the given substitution to find the exact value of the integral. Almost all showed complete working.

Answers: **(i)** $a = 4$ **(ii)** $\frac{16\pi}{3}$

Question 5

(i) There was evidence that most candidates knew that the result was true, some found it difficult to structure the simple proof needed.

(ii) It was important here to find the *corresponding* eigenvalues, but the candidates who used the characteristic equation to find the eigenvalues did not match them with their eigenvectors. Most used the matrix and the given eigenvectors correctly.

(iii) This part was also well done by most candidates. Some went on to find P^{-1} which was not required.

Answers: **(ii)** $\lambda = 3,\ 1,\ 2$ **(iii)** $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 7^3 & 0 & 0 \\ 0 & 6^3 & 0 \\ 0 & 0 & 3^3 \end{pmatrix} = \begin{pmatrix} 343 & 0 & 0 \\ 0 & 216 & 0 \\ 0 & 0 & 27 \end{pmatrix}$

Question 6

(i) Most candidates found the vertical asymptote. Some successfully attempted to find the oblique asymptote by division of polynomials.

(ii) Candidates who remembered that they needed to use the discriminant to ascertain whether a quadratic has two roots were generally successful, but others assumed that all quadratics have two roots.

(iii) Most candidates differentiated correctly and set the resulting quadratic to zero, but then assumed that this had two roots without reference to the discriminant. In fact, it has no roots, so there are no turning points. Some candidates misremembered the quotient rule for differentiation.

(iv) Sketches need to be carefully drawn to show behaviour near the asymptotes, so candidates must beware of curving away from the asymptotes. In general more care needs to be taken over labelling key points, and ensuring curves are the correct shape.

Answers: **(i)** $x = -1$ and $y = x + a - 1$ **(iii)** None

Question 7

(i) This question was tackled well by most candidates with few errors, though some did not show all the necessary working. Many candidates multiplied out $(1 - s^2)^3$ in preference to using a binomial expansion and errors did sometimes creep in here.

(ii) Although most candidates made the connection between the two parts, a significant number did not check to make sure that they wrote down six distinct roots, and that they eliminated any roots of $\sin(2\theta)$.

Answers: **(ii)** $\sin\frac{n\pi}{8}$ $n = \pm 1, \pm 2 \pm 3$ or $1, 2, 3, 9, 10, 11$

Question 8

(i) Most candidates remembered how to find the Cartesian equation of a plane, recognising the point on the plan as well as the two vectors lying in the plane from the given equation.

(ii) Apart from occasional arithmetic errors, most went on to find the angle between the two planes correctly. The most direct and most popular method of finding the line of intersection was by finding its direction using the vector product of the normal, and then working out a point on both lines. Some candidates solved the two equations simultaneously.

Answers: **(i)** $-x + 8y - 4z = 3$ **(ii)** $\theta = 72.5°$ **(iii)** e.g. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$

Question 9

(i) Although most candidates recalled the formula for the area of a sector and used the correct limits, a number did not integrate $\cot(\theta)$ correctly. This type of function is standard.

(ii) and (iii) Some candidates had difficulty with these parts, not appreciating the need to use $y = r\sin\theta$ for the distance from the initial line, and find where $\frac{dy}{d\theta} = 0$ to find the maximum distance of C from the initial line. Some candidates showed good insight in realising that **(iii)** could be found by considering the maximum value of $\sin(2\theta)$.

(iv) The curve was generally sketched well.

Answers: **(i)** $-\frac{25}{2}\ln\sin 0.01 \approx 57.6$ **(ii)** $y \approx 0.5$ **(iii)** $y = \frac{5\sqrt{2}}{2}$ $(= 3.54)$

Question 10

(i) Candidates were usually able to find the complementary function and particular integral with few problems. They were then able to use the initial conditions to complete their solution correctly. A small number of errors in using the wrong variable were seen.

(ii) Even candidates who recognised the limiting function did not all recall the method for writing $a\cos(t) + b\sin(t)$ as a single function involving sine despite the structure being given in the question.

Answers: (i) $x = e^{-t}(9\cos 3t + 2\sin 3t) - 6\cos 3t + \sin 3t$

Question 11

EITHER:

(i) and (ii) Those who tackled this option were usually able to use the difference method to prove the standard results given. Most handled the necessary algebra well, though a few left out vital steps in their working.

(iii) This was well done by most candidates.

(iv) Most candidates tackled this successfully by subtracting the sum of even terms from *S*. Some expanded $(2r-1)^3$ and summed the individual parts, though this needed more algebraic handling.

(v) This was well done by most candidates

Answers: **(iv)** $(N+1)^2\left(2N^2+4N+1\right)$ **(v)** 2

OR:

(i) Candidates approached this by substituting a value for either x or y. The resulting quadratic equation in x is a perfect square, the resulting cubic in y results in a linear factor, plus a quadratic without real roots.

(ii) and (iii) Most candidates used implicit differentiation accurately to find A_1 and A_2.

(iv) Most candidates were able to use integration by parts to find the reduction formula. Some candidates appeared to be confused by the notation used in the question.

(v) Many candidates, including those who had not been able to derive the reduction formula, were able to apply the formula successfully to find I_3.

Answers: **(v)** $I_3 = \frac{2}{5} + 6I_1$

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

Key messages

- Candidates should take great care when drawing graphs to make sure that their sketches show the correct behaviour near an asymptote. Sketches should be drawn and labelled with care.
- Candidates should ensure that they explain all steps in a solution, particularly when proving a given result.
- They should learn any standard formulae which are not on the formula sheet.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus, showed their working clearly, and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. It appeared that all were able to complete the paper in the time allowed, and a number of candidates had time to tackle both options for the final question. Work was generally well presented, though graph sketching was less satisfactory.

Comments on specific questions

Question 1

Most candidates were able to use the properties of roots to solve the first part accurately. Those who used the cubic to find $\sum\alpha^3$ were usually successful, though a small number forgot to find S_0. Errors were seen when candidates did not recall a formula for $\sum\alpha^3$ correctly.

Answers: **(i)** –1 **(ii)** –58

Question 2

Most candidates were able to find the corresponding eigenvalue in part **(i)**, and realised that they could read the eigenvalues from the diagonal form of the matrix for **(ii)**. Some candidates calculated the eigenvalues. Almost all were able to find the corresponding eigenvector either by using the vector product method, or by forming equations. The majority of candidates were able to calculate an eigenvalue of $A + A^6$ in **(iii)** and write down the correct eigenvalue, though some did not match the two. Some candidates wrote the matrix in diagonal form, which was not required.

Answers: **(i)** 2 **(ii)** –2 $\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ or equivalent **(iii)** 66, 62 or 2 $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$

Question 3

(i) Most candidates placed the loop in the correct position, though not all diagrams were labelled. Good sketches showed clearly the shape at the two extremities, and were carefully drawn to show the symmetry.

(ii) This part was well done, with most candidates remembering the formula correctly and using the double angle formula.

(iii) Better responses eliminated r and θ and simplified the resulting equation correctly. There were some errors particularly in the left hand side with incorrect powers of r appearing.

Answers: **(ii)** $\frac{\pi a^2}{12}$ **(iii)** $\left(x^2+y^2\right)^2 = ax\left(x^2-3y^2\right)$

Question 4

(i) This question was very well done by most candidates, with the majority of them remembering the correct form of the complementary function and the particular integral. Slips with mixing the variables were rare.

(ii) Most candidates recognised that the trigonometric function would form the approximate solution. Several went on to describe the oscillatory behaviour of this function.

Answers: **(i)** $x=\left(A+Bt\right)e^{-t}-2\cos t$ **(ii)** $x \approx -2\cos t$

Question 5

(i) Most candidates used reduced row techniques well, showing their working clearly, and were able to reduce the matrix and find the rank.

(ii) The majority of candidates went on to form equations and hence find a basis of the null space accurately, with very few arithmetic mistakes.

(iii) Better responses identified the particular solution from the second column of the matrix, and went on to reach the solution using the basis from part **(ii)**. Some candidates formed a set of equations and successfully solved them. A number of candidates did not understand the problem and gave the vector $\begin{pmatrix}2\\5\\8\\-2\end{pmatrix}$ as the particular solution.

Answers: **(i)** $r\left(\mathbf{M}\right)=4-2=2$ **(ii)** Two from $\begin{pmatrix}-2\\3\\3\\0\end{pmatrix}, \begin{pmatrix}1\\-3\\0\\3\end{pmatrix}, \begin{pmatrix}-1\\0\\3\\3\end{pmatrix}$ *or* $\begin{pmatrix}0\\-1\\1\\2\end{pmatrix}$ or equivalent

(iii) $\mathbf{x}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\3\\3\\0\end{pmatrix}+\mu\begin{pmatrix}1\\-3\\0\\3\end{pmatrix}$ or equivalent

Question 6

(i) Most candidates attempted to follow the structure of an induction proof, though sometimes wording was inaccurate: a common error is to assume that the result is true ‘for all integer values of k’ instead of for a particular value. Better responses showed full working, including justification of the two end terms as well as the general term. Most candidates were able to show the case when n = 1, and understood the notation.

Question 7

(i) Candidates expanded and simplified the expressions, and used the formula sheet to find the summations. Most showed complete working which is essential in questions where the answer is given. A few missed out steps in the simplification.

(ii) This part was also well done, with most candidates recognising the partial fractions, and cancelling to find the required formula.

(iii) Although many candidates were able to reduce the fraction to a polynomial, only stronger candidates referred to the fact that N is an integer to justify the final answer.

(iv) Few candidates were able to find the limit from their answer to **(ii)**.

Answers: **(iv)** 36

Question 8

(i) This question was well done by those candidates who were careful in their expansion and grouping, and used the correct substitution to reach the expression with multiple angles, taking care over the important factor of 2 in the formula.

(ii) Most candidates used their result from **(i)** and integrated directly, whilst a minority used a substitution instead. The processes of integration and of evaluation of the resulting expression were very well done, with exact answers being given as required.

Answers: **(i)** $\cos^6\theta = \frac{1}{32}(10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta)$ **(ii)** $\frac{1}{16}\left(5\pi + \frac{44}{3}\right)$

Question 9

(i) The first part of this question was well done, with most candidates recognising the horizontal asymptote either by division or considering what happens as $x \to \pm\infty$ or by division.

(ii) Most candidates tackled this second part well forming a quadratic in, and using the discriminant to determine the range of, y. Few candidates realised that they had to justify the strict upper inequality, and most of those did so by referring to the asymptote, though some elegant justifications referring to the structure of the function itself were also seen.

(iii) The question asked candidates to find stationary points, so they needed to differentiate; this was made easier when the function had been written in the form $5 - \mathrm{f}(x)$. Most candidates remembered to find both coordinates.

(iv) Most candidates found the y intercept and plotted the minimum point correctly. Better responses showed sketches representing the behaviour near the asymptote correctly, showing the curve clearly approaching the asymptote.

Answers: **(i)** $y = 5$ **(iii)** $x = -\frac{1}{2}, y = -\frac{1}{3}$

Question 10

(i) This part of the question was done well, with few slips.

(ii) Most candidates remembered the relevant formula for the shortest distance between two lines and applied it accurately. Some assumed that the common perpendicular went through one of the given points. Those who simplified the direction of the common perpendicular to $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ were usually able to come up with the correct numerical answer; others struggled to simplify the expression to eliminate m.

(iii) Very few errors were seen in part **(iii)**, with candidates knowing how to find the angle between two planes.

Answers: (ii) $\sqrt{2}$ (iii) $\theta = 12.2°$

Question 11

EITHER:

(i) Most candidates were able to find $\frac{dy}{dx}$ and then use the correct formula for the second derivative. Simplifying expressions early made the manipulation easier. The quotient rule was applied well.

(ii) The formula for the mean value was not used accurately in some cases, with candidates being unsure of the variables and the domain of x. Some candidates did realise that the required integral was simply $\frac{dy}{dx}$, and used the correct limits.

(iii) Most candidates knew the formula for the area of the surface, though a few omitted the y or placed it outside the integration. A small number of candidates left out the factor of 4 when finding $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ but the majority of candidates tackled this question very well.

(iii) $\frac{169\,984}{35}\pi\,(=15\,300)$

OR:

(i) Most candidates found the reduction formula, spotting the necessary split in the integrand.

(ii) The second part was also well done, although a few candidates did not show the complete working, including finding the new limits.

(iii) Candidates recognised that they needed to find I_3, and started by calculating either I_0 or I_1 correctly.

Answers: **(iii)** $I_3 = \frac{16 - 9\sqrt{2}}{35}$

FURTHER MATHEMATICS

Paper 9231/13
Paper 13

Key messages

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General comments

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Question 10

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(ii) Even candidates who recognised the limiting function did not all recall the method for writing $a\cos(t) + b\sin(t)$ as a single function involving sine despite the structure being given in the question.

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EITHER:

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Answers: **(iv)** $(N+1)^2\left(2N^2+4N+1\right)$ **(v)** 2

OR:

(i) Candidates approached this by substituting a value for either x or y. The resulting quadratic equation in x is a perfect square, the resulting cubic in y results in a linear factor, plus a quadratic without real roots.

(ii) and (iii) Most candidates used implicit differentiation accurately to find A_1 and A_2.

(iv) Most candidates were able to use integration by parts to find the reduction formula. Some candidates appeared to be confused by the notation used in the question.

(v) Many candidates, including those who had not been able to derive the reduction formula, were able to apply the formula successfully to find I_3.

Answers: **(v)** $I_3=\frac{2}{5}+6I_1$

FURTHER MATHEMATICS

Paper 9231/21
Paper 21

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

Non-standard symbols introduced by candidates should be defined or explained, for example by showing forces or velocities in a suitable diagram. Annotating a diagram printed in the question paper may be sufficient if the result is clear, but it may well be preferable to draw a fresh diagram within the answer.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Question 5** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many, but not all. When relevant in Mechanics questions it is helpful to show on a diagram what forces are acting and also their directions as in **Question 4** and the directions of motion of particles. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 9** and **10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances, for example.

Comments on specific questions

Question 1

Almost all candidates used the standard relationship $T = \dfrac{2\pi}{\omega}$ between the given period T of the motion and the unknown value of ω, here 4. This enables the required speed v to be found from the standard SHM formula $v = \omega\sqrt{\left(a^2 - x^2\right)}$ with $a = 3$ and $x = 1$.

Answer: $8\sqrt{2}$ or $11{\cdot}3\ \mathrm{ms^{-1}}$

Question 2

Many candidates were able to formulate two simultaneous equations for the speeds v_A and v_B of the two spheres after the collision by means of conservation of momentum and Newton's restitution equation. These may then be combined to verify the given expression for v_B and to find a similar expression for v_A, though several candidates overlooked the latter requirement. The required value of e follows from equating the speed of A after the collision, measured in the reverse direction, to $\dfrac{u}{2}$, with careful attention to the sign when doing so. The wrong sign will give a negative value of e, which should indicate to candidates that an error has been made. The value of e enables the speed of B after the collision, namely $\dfrac{7u}{4}$, to be found, and hence the ratio of the respective losses in kinetic energy of A and B. There is no need to evaluate the expression for v_A found in part **(i)**, as some candidates did, since it is already known to equal $\dfrac{u}{2}$ in magnitude.

Answers: **(i)** $\left(\dfrac{u}{7}\right)(1-6e)$ **(ii)** $\dfrac{3}{4}$ **(iii)** $2 : 1$

Question 3

Finding the required moment of inertia I presented most candidates with little difficulty, requiring the use of standard formulae and the parallel axis theorem to formulate and then sum the individual moments of inertia of the rod and disc. In the second part of the question an expression for the angular speed ω is first found by equating the rotational energy $\dfrac{1}{2}I\omega^2$ of the object to the change in potential energy as it rotates through an angle $\dfrac{\pi}{2} - \theta$. Equating this expression to the given angular speed yields a quadratic equation for x, and hence the two possible values of x. Care is needed in finding the change of potential energy; if candidates wish to treat the object as a body of mass $3M$, then the distance of its centre of mass from the axis must be found and used.

Answers: **(i)** $13\,Ma^2 + 2\,Mx^2$ **(ii)** a, $2a$

Question 4

As in all similar questions, candidates are well advised to first identify all the forces acting on the rod, preferably showing them on a diagram so that the symbols used to represent them are clear. This process reveals an unknown reaction acting on the rod at the hinge, suggesting that moments be taken about the hinge in order to produce an equation in which the only unknown is the required tension. Two perpendicular components of the reaction at the hinge may then be found by resolving the forces on the rod in two corresponding perpendicular directions. Candidates have the obvious choice of either horizontal and vertical directions, or parallel and normal to the rod, with the former choice probably a little simpler and certainly more popular. In either case, the two components then yield the magnitude and direction of the reaction at the hinge. Finally the modulus of elasticity λ follows from the use of Hooke's law $T = \dfrac{\lambda(CD - 2a)}{2a}$, using the value of T from part **(i)**, though a large number of candidates had difficulty in finding correctly the length $3a$ of CD.

Answers: **(ii)** 1.50 W at 33.7° above the horizontal **(iii)** $\frac{17W}{6}$

Question 5

Almost all candidates realised that AE may be verified by equating the equilibrium tensions in the two strings, with appropriate use of Hooke's law. An application of Newton's law at a general displacement x gives the product of m and the acceleration of the particle in terms of the difference in tensions in the two strings, from which the standard form of the SHM equation results, with $\omega^2 = 3.5\,\lambda$ here. Care is needed, particularly over signs, if errors are not to be introduced in deriving this result. Since the period is as usual $\frac{2\pi}{\omega}$, equating this to the given value $\frac{\pi}{7}$ gives the required value of λ.

Answer: **(iii)** 56

Question 6

As well as integrating $f(x)$ to find the distribution function $F(x)$, candidates should preferably state that this result holds true for $4 \leqslant x \leqslant 16$, that $F(x) = 0$ for $x < 4$ and that $F(x) = 1$ for $x > 16$. The first step in the second part is to find or state the distribution function of Y, and this is then differentiated to give the required probability density function.

Answers: **(i)** $\left(\frac{1}{40}\right)\left(x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + 8\right)\ (4 \leqslant x \leqslant 16)$ **(ii)** $\left(\frac{1}{40}\right)\ (3y^2 - 8)\ \ (2 \leqslant y \leqslant 4)$

Question 7

Having written down the probability density function of T, this may be integrated to find the distribution function $F(t)$ for $t \geqslant 0$. $P(T > 750)$ follows from $1 - F(750)$, and not of course $F(750)$. The method for finding the median value m of T by equating $F(t)$ to $\frac{1}{2}$ was well known.

Answers: **(i)** $0\,(t < 0)\quad \left(\frac{1}{500}\right)e^{-\frac{t}{500}}\quad (t \geqslant 0)$ **(ii)** 0.223 **(iii)** 500 ln 2 or 347 (hours)

Question 8

A key decision to be made is whether to use an estimate of a combined variance or a pooled estimate of a common population variance. In the absence of any information in the question about the variances of the two distributions, it is more appropriate to estimate and use a combined variance. However the alternative is acceptable to the examiners provided candidates state explicitly that they are assuming the two population variances to be equal. Most candidates knew how to conduct the test in principle. An unbiased estimate of 16.65 for the combined variance leads to a calculated z-value of 2.06. Since it is a two-tailed test, comparison with the tabulated value of 1.96 leads to acceptance of the alternative hypothesis, namely that $\mu_A \neq \mu_B$. The most common valid response to the second part of the question was that no assumption about normality is needed since the sample sizes are large, and hence the central limit theorem applies.

Question 9

As in all such tests, the hypotheses required in the first part should be stated in terms of the population mean and not the sample mean. The unbiased estimate 0.0052 of the population variance may be used to calculate a t-value of 1.37. Since it is a one-tail test, comparison with the tabulated value of 1.895 leads to acceptance of the null hypothesis, so the mean height of students is not greater than 1.70 metres. Finding the confidence interval was a straightforward task for many candidates, using the tabular value 2.365. The sample mean must be used as the centre of the confidence interval, and not 1.7.

Answer: **(ii)** [1.67, 1.80]

Question 10

Since the mean values of x and y satisfy the equation of the given regression line of y on x, the mean value of x and hence the sum Σx may be found by using the given mean value of y. The equation of the regression line of x on y follows from first finding the gradient and then utilising the values of the means. The required value of the correlation coefficient is found using the standard formula. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho > 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison of the coefficient value found earlier with the tabular value 0.549 leads to a conclusion of there being evidence of positive correlation.

Answers: **(i)** $x = 1.88y - 0.721$ **(ii)** 0.931.

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This optional question was attempted by a minority of candidates, and many of those who did so produced good attempts. In the first part they correctly verified the given equation by conservation of energy between A and C and also by finding the reactions at A and C and equating their ratio to the given value $\frac{8}{9}$. The second part firstly requires conservation of energy to find that the speed at B is $\sqrt{\left(\frac{4ag}{5}\right)}$, showing that P does reach the level of B. The normal reaction force on P as it reaches B is found to approach zero, so P does not (quite) lose contact with the inner surface of the sphere.

Question 11 (Statistics)

Candidates largely knew how to calculate the required estimate of the sample mean from $\sum \frac{x\mathrm{f}(x)}{200}$. In the second part, the expected frequency is verified by using the normal distribution function to find $200 \times P\left(\frac{(151-150)}{1.2} \leqslant Z < \frac{(152-150)}{1.2}\right)$. In the final part a clear statement of the null hypothesis, such as 'the normal distribution is a good fit to the results', is preferable to a more vague statement such as 'it fits'. Most candidates were apparently aware that in such questions cells may need to be combined in order to ensure that all the expected values are at least 5, and here the first and last two cells must be so combined. Indeed the goodness of fit test was often carried out well. Candidates should take particular care over the number of degrees of freedom when quoting the appropriate critical value, which is here 11.07. A comparison with the calculated value 15.6 of X^2 leads to acceptance of the alternative hypothesis, and hence the conclusion that the normal distribution does not fit the results.

Answer: **(i)** 150.3

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

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General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a very strong preference for the Statistics option, though some of the candidates who chose the Mechanics option produced good attempts. Indeed all questions were answered well by some candidates, with **Question 5** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, was seemingly heeded by many, but not all. When relevant in Mechanics questions it is helpful to show on a diagram what forces are acting and also their directions as in **Question 4** and the directions of motion of particles. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 9** and **11** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need to explain their working clearly in those questions which require certain given results to be verified rather than finding unknown results. It is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

The rubric for the paper specifies that non-exact numerical answers be given to 3 significant figures, and while most candidates abided by this requirement, a few rounded intermediate results to this same or lower accuracy with consequent risk that the final result is in error in the third significant figure. Such premature approximations should therefore be avoided, particularly when the difference of two numbers of similar sizes is to be taken, as can happen when taking the difference of means or estimating variances, for example.

Comments on specific questions

Question 1

Almost all candidates successfully used the standard SHM relationship $v^2 = \omega^2 (a^2 - x^2)$ to find the speeds, and hence the kinetic energy, at the two points A and B. Equating the ratio of these kinetic energies to the given value yields an equation in a^2, and the hence the required value of the amplitude a.

Answer: 0·7 m.

Question 2

Most candidates correctly formulated and solved two equations for the velocities of the spheres A and B after their first collision by means of conservation of momentum and Newton's restitution equation, and then combined these equations to give the required expressions for the speeds. Candidates found the second part somewhat more challenging, and a variety of different approaches to finding the required time t were seen. The first step is to note that the speed of B is halved after colliding with the wall. It is then possible to write down, for example, a single equation for the distance x from the wall of the second collision between the spheres by equating the time for A to travel $d - x$ to the sum of the times for B to travel d to the wall and then x back to meet A. This gives $x = \frac{d}{3}$, and hence t. Alternatively the distance $\frac{2d}{5}$ which A has travelled when B collides with the wall may be found, and then the times for the subsequent travel of A and B related. In all cases, it is helpful to the examiners, particularly when the required distance is found incorrectly, to give an adequate explanation of the approach being employed.

Answers: **(i)** $\frac{4u}{9}, \frac{10u}{9}$ **(ii)** $\frac{3d}{2u}$

Question 3

Many candidates were able to correctly formulate two equations involving the speed at B by using conservation of energy and a radial resolution of forces, with the tension equated to zero in the latter. Combining them yields the given expression for u^2. The maximum tension occurs of course at the lowest point of the motion, and it is found from $\frac{mV^2}{a} + mg$ where the speed V at this point follows from conservation of energy. Alternatively the tension may similarly be found at a general point, involving $\cos\theta$ where OP is at an angle θ to the downward vertical for example, and its maximum value clearly occurs when $\theta = 0$, so that $\cos\theta = 1$.

Answer: **(ii)** $\frac{75mg}{13}$

Question 4

Candidates are well advised to first identify all the forces acting on the rod, preferably showing them on a diagram in their answer book so that the symbols used to represent them are clear. This process reveals that three of the forces are unknown, namely the reaction R_A and friction F_A at A and the normal reaction R_C at C. A further unknown is the required distance $AC = x$, implying that four independent equations must be formulated in order to find x. One such follows immediately from the information about limiting equilibrium in the question, namely $F_A = \mu R_A$, and there are a variety of possible moment and force resolution equations to choose the other three from. Taking moments about C has the slight advantage that a single resolution along the rod will then suffice, since the reaction at C is not introduced, but taking moments about A was a popular choice. Combining the various equations produces the required expression for x. The required inequality for μ in part **(ii)** follows from $x \leqslant 2a$, and not from $F_A \leqslant \mu R_A$. In the final part, replacing x by $\frac{3a}{2}$ in the expression previously found for μ yields the value of μ and hence the magnitude of the resultant force at A.

Answers: **(i)** $\frac{a(1+\mu)}{2\mu}$ **(iii)** $\frac{1}{2}$ $\left(\frac{\sqrt{5}}{3}\right)W$

Question 5

The key to finding the required moment of inertia I is to not only use standard formulae and the parallel axis theorem to formulate and then sum the individual moments of inertia of the rod and disc, but also to realise that the specified axis is in the plane of the object rather than perpendicular to it. This necessitates use of the perpendicular axis theorem for the disc, so that its moment of inertia about the axis is 78 Ma^2. As in all such questions where a given result must be shown, candidates should include sufficient working so as to justify full credit. Candidates who simply write down a sum of terms with no explanation whatever run considerable risk, since an error in only one term can cast doubt on the validity of their whole process and thereby lose credit. In the second part, the couple acting on the system should be found in terms of $\sin\theta$, where θ is the small angular displacement, and equated to $I\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$. Approximating $\sin\theta$ by θ yields the familiar form $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega^2\theta$ of the standard SHM equation, in which the minus sign is essential. An expression for the period in terms of k is then obtained from $\dfrac{2\pi}{\omega}$, and equating this expression to the given period yields the required value of k. Candidates should be very much aware that the constant parameter ω in the SHM equation is not equal to $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ even though the same symbol is sometimes used to denote the latter, and thus finding $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$ from conservation of energy at some arbitrary angular displacement and equating it to the SHM parameter ω is wholly invalid.

Answer: **(ii)** 6

Question 6

Calculating the confidence interval rarely presented problems for candidates, who found the sample mean to be $\dfrac{90.3}{8}$ and an unbiased estimate of the population variance to be 3·487. They should take care to select the appropriate critical t-value, here 2·365.

Answer: [9·7, 12·8[5]]

Question 7

The first step is to find or state the distribution function G of Y over $0 \leqslant y \leqslant 9$, namely $\dfrac{(y + y^2)}{90}$, which is then differentiated to obtain the probability density function g(y). The method for finding the required mean value of Y was generally known, namely integration of y g(y), though the mean was sometimes confused with the median.

Answers: **(i)** $\dfrac{(1+2y)}{90}$ for $0 \leqslant y \leqslant 9$, 0 otherwise **(ii)** 5·85

Question 8

Most candidates were able to verify the given value of p by using $q^4 = 0{\cdot}4096$, where $q = 1 - p$. The required probability in the second part is readily found from $q^5 p$. The starting point in the final part is to formulate the inequality $1 - q^N > 0{\cdot}9$, solution of which gives $N > 10{\cdot}3$ and hence the least integral value of N. It is important to use the correct power of q in the initial inequality, and to reverse the inequality when multiplying through by a negative factor. The question specifies that Lan catches the bus on five days each week, so candidates should identify the day and week corresponding to their least integer N using a week of 5 rather than 7 days.

Answers: **(ii)** 0·0655 **(iii)** 11, Monday of third week

Question 9

Most candidates stated the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho \neq 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison of the magnitude 0·7214 of the given coefficient value r with the tabular value 0·878 leads to a conclusion of there being no evidence of non-zero correlation. Since the mean values of x and y satisfy both given regression line equations, these may be combined to give an equation relating c and d, while a second such equation follows from equating cd to r^2. In the final part, x may be estimated from the regression line of x on y, and a variety of acceptable comments may be made on the reliability of this estimate.

Answers: **(ii)** –0·294, –1·77 **(iii)** 4·60[5]

Question 10

Candidates largely knew how to calculate the mean and variance of the sample, though in the latter case some chose to divide by 40 and others by 39. An argument can be made for either choice, so both are acceptable. An appropriate explanation regarding the Poisson distribution is that the mean and variance values are similar. In the second part, it is sufficient to state $\frac{40 \times \lambda^4 e^{-\lambda}}{4}!$ with $\lambda = 2{\cdot}95$, for example, and most candidates did so. In the final part a clear statement of the null hypothesis, such as ‘the Poisson distribution is a good fit to the data’, is preferable to a more vague statement such as ‘it fits’. Many candidates were apparently aware that in such questions cells may need to be combined in order to ensure that all the expected values are at least 5, and here the first two cells must be so combined, as must the last three. Apart from this, the goodness of fit test was often carried out well. Candidates should take particular care over the number of degrees of freedom when quoting the appropriate critical value. Because the question implies that the parameter 2·95 has been determined from the observational data, 5 cells correspond to 3 degrees of freedom here, giving a critical value 7·815. A comparison with the calculated value 1·20 of X^2 leads to acceptance of the null hypothesis, and hence the conclusion that the Poisson distribution is a good fit to the data.

Answer: **(i)** 2·65 or 2·72

Question 11 (Mechanics)

This optional question was attempted by a minority of candidates, and many of those who did so produced good attempts. After the M kg particle is detached, Newton’s second law of motion can be applied to the remaining particle to derive eventually an equation in the form $\frac{d^2x}{dt^2} = -\omega^2 x$ with $\omega^2 = 25$, thus verifying simple harmonic motion. Achieving this standard form of the SHM equation requires x to be the displacement from the centre of the oscillations, which is of course the equilibrium position for the 2 kg particle, found using Hooke’s law. The period is verified using the standard formula $\frac{2\pi}{\omega}$. The amplitude a is found from the standard SHM formula $v^2 = \omega^2(a^2 - x^2)$ with $v = 0{\cdot}4$ and $x = 0{\cdot}06$. This amplitude is of course equal to the difference between the equilibrium positions before and after the M kg particle is detached, enabling M to be deduced after a further application of Hooke’s law at the former equilibrium position.

Answers: **(i)** 1·2 m **(ii)** 0·1 m **(iii)** 0·5

Question 11 (Statistics)

As in all such tests, the hypotheses should be stated in terms of the population means and not the sample means. Most candidates found an unbiased pooled estimate of 0·04882 for the common variance. This enables the value 0·773 to be found for t, and comparison with the tabulated value of 1·337 leads to acceptance of the null hypothesis, and hence a conclusion of the scientist's claim not being justified. Since the question states that the two population variances should be assumed to be equal, it is inappropriate to base the test on an estimate of the variance of the combined populations, as a few candidates did. In the second part an expression for t in terms of p is found using a sample mean of 1·28 and an estimated variance of 0·042. Since this must be greater than the tabulated value 1·415, the greatest possible value of p may then be found. Rounding the limiting value to 3 significant figures in accordance with the rubric gives 1·18 rather than 1·17.

Answer: **(ii)** 1·18

FURTHER MATHEMATICS

Paper 9231/23
Paper 23

Key messages

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Comments on specific questions

Question 1

Almost all candidates used the standard relationship $T = \dfrac{2\pi}{\omega}$ between the given period T of the motion and the unknown value of ω, here 4. This enables the required speed v to be found from the standard SHM formula $v = \omega\sqrt{(a^2 - x^2)}$ with $a = 3$ and $x = 1$.

Answer: $8\sqrt{2}$ or $11{\cdot}3\ \text{ms}^{-1}$

Question 2

Many candidates were able to formulate two simultaneous equations for the speeds v_A and v_B of the two spheres after the collision by means of conservation of momentum and Newton's restitution equation. These may then be combined to verify the given expression for v_B and to find a similar expression for v_A, though several candidates overlooked the latter requirement. The required value of e follows from equating the speed of A after the collision, measured in the reverse direction, to $\dfrac{u}{2}$, with careful attention to the sign when doing so. The wrong sign will give a negative value of e, which should indicate to candidates that an error has been made. The value of e enables the speed of B after the collision, namely $\dfrac{7u}{4}$, to be found, and hence the ratio of the respective losses in kinetic energy of A and B. There is no need to evaluate the expression for v_A found in part **(i)**, as some candidates did, since it is already known to equal $\dfrac{u}{2}$ in magnitude.

Answers: **(i)** $\left(\dfrac{u}{7}\right)(1-6e)$ **(ii)** $\dfrac{3}{4}$ **(iii)** 2 : 1

Question 3

Finding the required moment of inertia I presented most candidates with little difficulty, requiring the use of standard formulae and the parallel axis theorem to formulate and then sum the individual moments of inertia of the rod and disc. In the second part of the question an expression for the angular speed ω is first found by equating the rotational energy $\dfrac{1}{2}I\omega^2$ of the object to the change in potential energy as it rotates through an angle $\dfrac{\pi}{2} - \theta$. Equating this expression to the given angular speed yields a quadratic equation for x, and hence the two possible values of x. Care is needed in finding the change of potential energy; if candidates wish to treat the object as a body of mass $3M$, then the distance of its centre of mass from the axis must be found and used.

Answers: **(i)** $13\,Ma^2 + 2\,Mx^2$ **(ii)** a, $2a$

Question 4

As in all similar questions, candidates are well advised to first identify all the forces acting on the rod, preferably showing them on a diagram so that the symbols used to represent them are clear. This process reveals an unknown reaction acting on the rod at the hinge, suggesting that moments be taken about the hinge in order to produce an equation in which the only unknown is the required tension. Two perpendicular components of the reaction at the hinge may then be found by resolving the forces on the rod in two corresponding perpendicular directions. Candidates have the obvious choice of either horizontal and vertical directions, or parallel and normal to the rod, with the former choice probably a little simpler and certainly more popular. In either case, the two components then yield the magnitude and direction of the reaction at the hinge. Finally the modulus of elasticity λ follows from the use of Hooke's law $T = \dfrac{\lambda(CD - 2a)}{2a}$, using the value of T from part **(i)**, though a large number of candidates had difficulty in finding correctly the length $3a$ of CD.

Answers: **(ii)** 1.50 W at 33.7° above the horizontal **(iii)** $\frac{17W}{6}$

Question 5

Almost all candidates realised that AE may be verified by equating the equilibrium tensions in the two strings, with appropriate use of Hooke's law. An application of Newton's law at a general displacement x gives the product of m and the acceleration of the particle in terms of the difference in tensions in the two strings, from which the standard form of the SHM equation results, with $\omega^2 = 3.5\,\lambda$ here. Care is needed, particularly over signs, if errors are not to be introduced in deriving this result. Since the period is as usual $\frac{2\pi}{\omega}$, equating this to the given value $\frac{\pi}{7}$ gives the required value of λ.

Answer: **(iii)** 56

Question 6

As well as integrating $f(x)$ to find the distribution function $F(x)$, candidates should preferably state that this result holds true for $4 \leqslant x \leqslant 16$, that $F(x) = 0$ for $x < 4$ and that $F(x) = 1$ for $x > 16$. The first step in the second part is to find or state the distribution function of Y, and this is then differentiated to give the required probability density function.

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Answer: **(ii)** [1.67, 1.80]

Question 10

Since the mean values of x and y satisfy the equation of the given regression line of y on x, the mean value of x and hence the sum Σx may be found by using the given mean value of y. The equation of the regression line of x on y follows from first finding the gradient and then utilising the values of the means. The required value of the correlation coefficient is found using the standard formula. Most candidates went on to state the null and alternative hypotheses correctly, which should be in the form $\rho = 0$ and $\rho > 0$, though some wrongly stated them in terms of r which conventionally relates to the sample and not the population. Comparison of the coefficient value found earlier with the tabular value 0.549 leads to a conclusion of there being evidence of positive correlation.

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Answer: **(i)** 150.3