CAMBRIDGE INTERNATIONAL EXAMINATIONS

Cambridge International Advanced Level

MARK SCHEME for the October/November 2014 series

9231 FURTHER MATHEMATICS

9231/21 Paper 2, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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International Examinations

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously 'correct' answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF	Any Equivalent Form (of answer is equally acceptable)
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
BOD	Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)
CAO	Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)
CWO	Correct Working Only – often written by a ‘fortuitous' answer
ISW	Ignore Subsequent Working
MR	Misread
PA	Premature Approximation (resulting in basically correct work that is insufficiently accurate)
SOS	See Other Solution (the candidate makes a better attempt at the same question)
SR	Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1	A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become ‘follow through $\sqrt{}$’ marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.
PA –1	This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question Number	Mark Scheme Details			Part Mark	Total
1	Use conservation of momentum, e.g.:	$2mv_A + mv_B = 8mu - 3mu$	B1		
	Use restitution (must be consistent with prev. eqn.):	$v_A - v_B = -e(4u + 3u)$	B1		
	Solve for v_A (or $3v_A$):	$v_A = \frac{1}{3}(5-7e)u$ $[v_B = \frac{1}{3}(5+14e)u\ ]$	M1		
	Find lower limit on e for which $v_A < 0$:	$5 - 7e < 0,\ e > \frac{5}{7}$ *or* $0{\cdot}714$	M1 A1	5	**[5]**
2	Find speed component along barrier:	$V\cos\beta = 4\cos\alpha$	B1		
	Find speed component normal to barrier:	$V\sin\beta = 0{\cdot}4 \times 4\sin\alpha$	B1		
	Find β by eliminating α with $V = 2$:	$V^2 = 2^2 = 1{\cdot}6^2\sin^2\alpha + 16\cos^2\alpha$ $1 - \sin^2\alpha + 0{\cdot}16\sin^2\alpha = 0{\cdot}25$ $\sin^2\alpha = \frac{0.75}{0.84} = \frac{25}{28} = 0{\cdot}8929$ <u>or</u> $\cos^2\alpha = \frac{3}{28} = 0{\cdot}1071$	M1		
		$\alpha = 1{\cdot}24$ rad *or* $70{\cdot}9°$	M1 A1	5	**[5]**
3	Use conservation of energy:	$\frac{1}{2}mv_B^2 = \frac{1}{2}mu^2 + 2mga\cos\alpha$ $[v_B^2 = u^2 + \frac{12ag}{5}\ ]$	B1		
	Use $F = ma$ radially at A and B (B1 for either):	$R_A = \frac{mu^2}{a} - mg\cos\alpha$ $R_B = \frac{mv_B^2}{a} + mg\cos\alpha$	B1		
	Equate R_B to 10 R_A:	$\frac{mv_B^2}{a} + mg\cos\alpha = 10\left(\frac{mu^2}{a} - mg\cos\alpha\right)$	M1 A1		
	Eliminate v_B^2:	$u^2 + 4ag\cos\alpha = 10u^2 - 11ag\cos\alpha$ $[\ v_B^2 = \frac{17ag}{5}\ ]$			
		$u^2 = (\frac{5ag}{3})\cos\alpha = ag$ **A.G.**	M1 A1	6	

	Use conservation of energy for loss of contact:	$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mga(\cos\alpha - \cos\theta)$ *or* $\frac{1}{2}mv_B^2 - mga(\cos\alpha + \cos\theta)$ B1		
	Use $F = ma$ radially with $R = 0$:	$\frac{mv^2}{a} = mg\cos\theta$ B1		
	Eliminate v^2 with $u^2 = ag$ to find $\cos\theta$:	$ga + 2ga(\cos\alpha - \cos\theta) = ga\cos\theta$ $\cos\theta = \frac{1}{3}(2\cos\alpha + 1) = \frac{11}{15}$ M1 A1	4	**[10]**
4 (i)	Relate F_A and R_A using $F = \mu R$:	$F_A = \frac{1}{3}R_A$ B1		
	Resolve horizontally:	$R_B = F_A$ [$= \frac{1}{3}R_A$] B1		
	Resolve vertically (may not be needed here):	$S = mg - R_A$ B1		
	EITHER: Take moments about C:	$R_B \frac{1}{4}l\sin\alpha + F_A \frac{3}{4}l\sin\alpha$ $+ mg\frac{1}{4}l\cos\alpha = R_A\frac{3}{4}l\cos\alpha$ M1 A1		
	Combine, using $\tan\alpha = \frac{3}{4}$, to find R_A:	$R_A + 3R_A + 4mg = 12R_A$ $R_A = \frac{1}{2}mg$ **A.G.** M1 A1		
	OR: Take moments about A:	$R_B l\sin\alpha + S\frac{3}{4}l\cos\alpha$ $= mg\frac{1}{2}l\cos\alpha$ (M1 A1)		
	Combine, using $\tan\alpha = \frac{3}{4}$, to find R_A:	$R_A + 3(mg - R_A) = 2mg$ $R_A = \frac{1}{2}mg$ **A.G.** (M1 A1)		
	OR: Take moments about B:	$F_A l\sin\alpha + mg\frac{1}{2}l\cos\alpha$ $= R_A l\cos\alpha + S\frac{1}{4}l\cos\alpha$ (M1 A1)		

	Combine, using $\tan\alpha = \frac{3}{4}$, to find R_A: $R_A + 2mg = 4R_A + mg - R_A$			
	$R_A = \frac{1}{2}mg$ **A.G.**	(M1 A1)		
	OR: Take moments about D: $R_A \frac{3}{4} l \cos\alpha$			
	$= R_B l \sin\alpha + mg \frac{1}{4} l \cos\alpha$	(M1 A1)		
	Combine, using $\tan\alpha = \frac{3}{4}$, to find R_A: $3R_A = R_A + mg$			
	$R_A = \frac{1}{2}mg$ **A.G.**	(M1 A1)	7	
(ii)	Use Hooke's Law to relate extn. e and nat. length L:			
	$S = \frac{1}{2}mg = \frac{2mge}{L}$, $e = \frac{1}{4}L$	B1		
	Find length of CD: $CD = \frac{3}{4} l \sin\alpha = \frac{9l}{20}$	B1		
	Combine to find L: $L - \frac{1}{4}L = \frac{9l}{20}$, $L = \frac{3l}{5}$	M1 A1	4	**[11]**

5 (i)	Find extn. of either string by equating equil. tensions:	$\dfrac{6mge_A}{3a} = \dfrac{mg(3a-e_A)}{2a}$ *or* $\dfrac{6mg(3a-e_B)}{3a} = \dfrac{mge_B}{2a}$ M1 A1 $e_A = \dfrac{3a}{5}$ *or* $e_B = \dfrac{12a}{5}$ A1		
	Find AO: **A.G.**	$AO = 3a + e_A$ *or* $6a - e_B = 3{\cdot}6a$ B1	4	
(ii)	Apply Newton's law at general point, e.g.: (lose A1 for each incorrect term)	$m\dfrac{d^2x}{dt^2} = \dfrac{mg(3a-e_A-x)}{2a} - \dfrac{6mg(e_A+x)}{3a}$ *or* $m\dfrac{d^2y}{dt^2} = -\dfrac{mg(3a-e_A+y)}{2a} + \dfrac{6mg(e_A-y)}{3a}$ M1 A2		
	Simplify to give standard SHM eqn, e.g.: **S.R.**: B1 if no derivation (max 3/6)	$\dfrac{d^2x}{dt^2} = -\dfrac{5gx}{2a}$ A1		
	Find period T using SHM with $\omega = \sqrt{\left(\dfrac{5g}{2a}\right)}$:	$T = 2\pi\sqrt{\left(\dfrac{2a}{5g}\right)}$ (A.E.F.) M1 A1	6	
(iii)	Find max speed using ωA with $A = 0{\cdot}2\ a$:	$v_{max} = \sqrt{\left(\dfrac{5g}{2a}\right)} \times 0{\cdot}2a$ $= \sqrt{\left(\dfrac{ag}{10}\right)}$ *or* $\sqrt{a}$ (A.E.F.) M1 A1	2	[12]
6	Estimate population variance for combined sample:	$s^2 = \dfrac{s_X{}^2}{50} + \dfrac{s_Y{}^2}{60}$ $= \dfrac{1391}{1500}$ *or* $0{\cdot}9273$ *or* $0{\cdot}9630^2$ M1		
	Calculate value of z (to 2 d.p., either sign):	$z = \dfrac{1.8}{s} = 1{\cdot}869$ M1 A1		
	Find $\Phi(z)$ and set of possible values of α (to 1 d.p.): (M1 A0 for $\alpha \leqslant 3{\cdot}1$ *or* $\alpha > 93{\cdot}8$)	$\Phi(z) = 0{\cdot}9692$ [*or* $96{\cdot}92\%$] $\alpha \leqslant$ (*or* $<$) $6{\cdot}2$ (allow $6{\cdot}1$) M1 A1	5	[5]

	S.R. Assuming equal population variances:	Explicit assumption (B1)		
	Find pooled estimate of common variance s^2	$\dfrac{(49s_X{}^2 + 59s_Y{}^2)}{108}$		
		$= \dfrac{2777}{108}$ *or* 25·71 *or* $5{\cdot}071^2$		
	and calculate value of z (to 2 d.p.):	$z = \dfrac{1.8}{s\sqrt{50^{-1} + 60^{-1}}} = 1{\cdot}854$ (M1 A1)		
	Find $\Phi(z)$ and values of α (to 1 d.p.): (M1 A0 for $\alpha \leqslant 3{\cdot}2$ *or* $\alpha > 93{\cdot}6$)	$\Phi(z) = 0{\cdot}9681$ [*or* 96·81%] $\alpha \leqslant$ (*or* <) 6·4 (M1 A1)		
7 (i)	State or find E(T):	$E(T) = \dfrac{1}{0.01} = 100$ B1	1	
(ii)	State or use eqn. for median m of T:	$\left[-e^{-0.01t}\right]_0^m = \dfrac{1}{2}$ (A.E.F.) M1		
	Find value of m:	$e^{-0.01m} = \dfrac{1}{2}$, $m = 100\ln 2 = 69{\cdot}3$ M1 A1	3	
	Find $P(T > 20)$:	$P(T > 20) = 1 - (1 - e^{-0.2})$		
	S.R. B1 for 0·181	$= e^{-0.2}$ *or* 0·819 M1 A1	2	**[6]**
8	Find mean of sample data for use in Poisson distn.:	$\lambda = \dfrac{225}{100} = 2{\cdot}25$ B1		
	State (at least) null hypothesis (A.E.F.):	H_0: Poisson distn. fits data B1		
	Find expected values $\dfrac{100\lambda^r e^{-\lambda}}{r!}$ (to 1 d.p.): (ignore incorrect final value here for M1)	10·540 23·715 26·679 20·009 11·255 5·065 1·899 0·6105 0·2275 M1 A1		
	Combine last four cells so that exp. value ⩾ 5:	O_i: . . . 16 14 4 E_i: . . . 20·009 11·255 7·802 *M1		
	Calculate value of χ^2 (to 1 d.p.; A1 dep *M1):	$\chi^2 = 1{\cdot}189 + 0{\cdot}582 + 5{\cdot}690 + 0{\cdot}803$ $+ 0{\cdot}6695 + 1{\cdot}853$ $= 10{\cdot}8$ (allow 10·7) M1 A1		
	State or use consistent tabular value (to 1 d.p.):	$\chi_{4,\,0.975}{}^2 = 11{\cdot}14$ (if cells combined) B1 [$\chi_{7,\,0.975}{}^2 = 16{\cdot}01$, $\chi_{5,\,0.975}{}^2 = 12{\cdot}83$]		
	Consistent conclusion (A.E.F., √ on two χ^2 values):	$\chi^2 < 11{\cdot}1$ so Poisson distn. fits B1√	9	**[9]**

9 (i)	Calculate gradient b in $y-\overline{y}=b\left(x-\overline{x}\right)$:	$S_{xy} = 513 - 73 \times \frac{64}{10} = 45{\cdot}8$ $S_{xx} = 651 - \frac{73^2}{10} = 118{\cdot}1$ $b = \frac{S_{xy}}{S_{xx}} = 0{\cdot}388$	M1 A1		
	Find regression line of y on x:	$y = \frac{64}{10} + 0{\cdot}388\ (x - \frac{73}{10})$	M1		
		$= 0{\cdot}388\,x + 3{\cdot}57$ *or* $\frac{(458x+4215)}{1181}$	A1	4	
(ii)	Find correlation coefficient r:	$S_{yy} = 462 - \frac{64^2}{10} = 52{\cdot}4$ $r = \frac{S_{xy}}{\sqrt{\left(S_{xx}S_{yy}\right)}} = 0{\cdot}582$	M1 A1	2	
(iii)	Find y when $x = 10$:	$y = 7{\cdot}45$	B1		
	State valid comment on reliability, e.g.:	Not reliable as value of r is small *or* reliable since $x = 10$ is in range *or* is near mean	B1	2	
(iv)	Formulate condition for N:	Require one-tail $r_{N,1\%} < r$ [0·582]	M1		
	Identify critical value near r using table:	15 *or* 16 (√ on r)	A1√		
	State set of possible values of N:	$N \geqslant 16$	A1	3	[11]

10	Find F(x) for $1 \leqslant x \leqslant 3$:	$F(x) = \frac{1}{2}(x - 1)$	B1		
	Find G(y) from $Y = X^3$ for $1 \leqslant x \leqslant 3$:	$G(y) = P(Y < y) = P(X^3 < y)$ $= P(X < y^{\frac{1}{3}}) = F(y^{\frac{1}{3}})$			
	(result may be stated)	$= \frac{1}{2}(y^{\frac{1}{3}} - 1)$; $(1 \leqslant y \leqslant 27)$	M1 A1; B1		
	State G(y) for other values of y:	0 $(y < 1)$ _and_ 1 $(y > 27)$	B1	5	
	Find g(y) for $1 \leqslant y \leqslant 27$ (√ on G(y)):	$g(y) = \frac{y^{-\frac{2}{3}}}{6}$ _or_ $\frac{1}{6y^{\frac{2}{3}}}$	B1√		
	Sketch g(y) for $1 \leqslant y \leqslant 27$ with g(y) = 0 on either side of this interval		B1 B1	3	
	Find mean of Y:	$E(Y) = \int y\, g(y)\, dy = \int (\frac{y^{\frac{1}{3}}}{6})\, dy$			
	(no need to find median = 8)	$= \left[\frac{y^{\frac{4}{3}}}{8}\right]_1^{27} = 10$	M1 A1		
	Find probability Y lies between median and mean:				
		$G(10) - G(8)$ _or_ $\lvert G(10) - \frac{1}{2} \rvert$			
		$= \frac{1}{2}\left(10^{\frac{1}{3}} - 8^{\frac{1}{3}}\right)$			
	(2 s.f. sufficient)	_or_ $\left\lvert \frac{1}{2}\left(10^{\frac{1}{3}} - 1\right) - \frac{1}{2} \right\rvert = 0{\cdot}077$ [2]			
			M1 A1	4	**[12]**

Question	Answer	Marks	Part marks	Total
11a (i)	*EITHER:* State or find (by ⊥ axes) MI of X about AB: $I_X = \frac{1}{2}mr^2$	M1 A1		
	State or find MI of Y (or Z) about AB: $I_Y = \frac{1}{2}mr^2 + mr^2 = \frac{3mr^2}{2}$	M1 A1		
	State or find (by ⊥ axes) MI of W about AB: $I_W = \frac{1}{2}\,3mR^2 = \frac{1}{2}\,3mr^2(1 + \frac{2}{3}\sqrt{3})^2$ $= \frac{1}{2}(7 + 4\sqrt{3})\,mr^2$	M1 A1		
	Find MI of object about AB: $I = (\frac{1}{2} + 2\times\frac{3}{2} + \frac{7}{2} + 2\sqrt{3})\,mr^2$ $= (7 + 2\sqrt{3})\,mr^2$ **A.G.**	M1 A1		
	OR: State or find MI of X, Y or Z about centre O: $I_X = mr^2 + m\left(\frac{2r}{\sqrt{3}}\right)^2 = \frac{7mr^2}{3}$	(M1 A1)		
	State or find MI of W about O: $I_W = 3mR^2 = 3mr^2(1 + \frac{2}{3}\sqrt{3})^2$ $= (7 + 4\sqrt{3})\,mr^2$	(M1 A1)		
	Find MI of object about O: $I_O = 3I_X + I_W = (14 + 4\sqrt{3})\,mr^2$	(M1 A1)		
	Find (by ⊥ axes) MI of object about AB: $I = \frac{1}{2}I_O = (7 + 2\sqrt{3})\,mr^2$ **A.G.**	(M1 A1)	8	
(ii)	Find new MI of object plus particle about AB: $I' = I + 9mR^2$ $= I + 3(7 + 4\sqrt{3})\,mr^2$ $= 14(2 + \sqrt{3})\,mr^2$	M1 A1		
	Find eqn for angular speed ω using energy: $\frac{1}{2}I'\omega^2 = 9mg \times R\sin 60°$	M1 A1		
	Substitute and simplify to find ω: $\omega^2 = \frac{9mgR\sqrt{3}}{I'}$			
	(AEF) $\omega = \sqrt{\frac{9g}{14r}}$ <u>or</u> $0{\cdot}802\sqrt{\frac{g}{r}}$	M1 A1	6	[14]

11b	Estimate population variance using A's sample:	$s_A^2 = \dfrac{\left(1422.34 - \frac{106^2}{8}\right)}{7}$			
	(allow use of biased: $\sigma_{A,8}^2 = 2{\cdot}23$ *or* $1{\cdot}493^2$)	$= \dfrac{446}{175}$ <u>or</u> $2{\cdot}549$ <u>or</u> $1{\cdot}596^2$	M1 A1		
	Find confidence interval:	$\dfrac{106}{8} \pm t\sqrt{\left(\dfrac{s_X^{\ 2}}{8}\right)}$	M1		
	State or use correct tabular value of t:	$t_{7,0.975} = 2{\cdot}36$ [5]	A1		
	Evaluate C.I. correct to 1 d.p.:	$13{\cdot}25 \pm 1{\cdot}335$ <u>or</u> $[11{\cdot}9, 14{\cdot}6]$	A1	5	
	State suitable assumptions (A.E.F.):	Distribution of B is Normal with same population variance	B1		
	State hypotheses (B0 for $\overline{a}$ …) e.g.:	H_0: $\mu_A = \mu_B$, H_1: $\mu_A > \mu_B$	B1		
	Estimate (or imply) B's population variance: (allow use of biased: $\sigma_{B,6}^2 = 1{\cdot}899$ <u>or</u> $1{\cdot}378^2$)	$s_B^2 = \dfrac{\left(971.53 - \frac{75.9^2}{6}\right)}{5}$ $= 2{\cdot}279$ <u>or</u> $1{\cdot}510^2$			
	and find pooled estimate of common variance s^2:	$s^2 = \dfrac{\left(7s_A^{\ 2} + 5s_B^{\ 2}\right)}{12}$ $= \dfrac{(17.84 + 11.395)}{12}$ $= 2{\cdot}436$ <u>or</u> $1{\cdot}561^2$	M1		
	Calculate value of t (to 2 d.p., either sign):	$t = \dfrac{(13.25 - 12.65)}{s\sqrt{\left(8^{-1} + 6^{-1}\right)}}$ $= \dfrac{0.6}{0.8430} = 0{\cdot}712$	M1 A1		
	State or use correct tabular t-value (to 2 d.p.): (or compare $0{\cdot}6$ with $1{\cdot}782\ s\sqrt{(8^{-1} + 6^{-1})} = 1{\cdot}50$)	$t_{12,0.95} = 1{\cdot}782$	B1		
	Consistent conclusion (AEF, √ on two t values):	[Accept H_0]; mean lengths are the same	B1√	7	
	Find confidence interval for the difference:	$13{\cdot}25 - 12{\cdot}65 \pm t\,s\sqrt{(8^{-1} + 6^{-1})}$	M1		
	Evaluate C.I. with $t_{12,0.975} = 2{\cdot}179$, to 2 d.p.:	$0{\cdot}6 \pm 1{\cdot}84$ <u>or</u> $[-1{\cdot}24, 2{\cdot}44]$	A1	2	**[14]**