CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the May/June 2014 series

9231 FURTHER MATHEMATICS

9231/12 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2014 series for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level components and some Ordinary Level components.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qn & Part	Solution	Marks
1	Let roots be α, α^{-1} and β $\Rightarrow \alpha + \alpha^{-1} + \beta = 0$ (1) Any of three for M1 Product of roots $\Rightarrow \beta = -q$ (2) A1 for another. Sum of products in pairs $\Rightarrow 1 + \beta(\alpha + \alpha^{-1}) = p$ (3) A1 for a third From (1) and (3) $1 - \beta^2 = p$ Wrong sign in (2) scores Using (2) $1 - q^2 = p$ or $p + q^2 = 1$ M1A0A1M1A0	M1 A1 A1 M1 A1 [5]
2	$(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$ $(n+1)^4 - 1^4 = 4\Sigma_{r=1}^{n} r^3 + 6\Sigma_{r=1}^{n} r^2 + 4\Sigma_{r=1}^{n} r + n$ $n^4 + 4n^3 + 6n^2 + 4n = 4\Sigma_{r=1}^{n} r^3 + n(2n^2 + 3n + 1) + 2n^2 + 2n + n$ $\Rightarrow \ldots \Rightarrow \Sigma_{r=1}^{n} r^3 = \frac{1}{4} n^2(n+1)^2$. (AG)	B1 [1] M1 A1A1 A1 [4]
3	H_k: $f(k) = 11^{2k} + 25^k + 22 = 24\lambda$ $f(0) = 1 + 1 + 22 = 24 = 1 \times 24$ $\Rightarrow H_0$ is true. $f(k+1) - f(k) = 11^{2k+2} + 25^{k+1} + 22 - (11^{2k} + 25^k + 22)$ $= 11^{2k}(121 - 1) + 25^k(25 - 1)$ $= 11^{2k} \times 24 \times 5 + 25^k \times 24 = 24\mu$ **Alternatively:** $f(k+1) = 11^{2k+2} + 25^{k+1} + 22$ $= 121.11^{2k} + 25.25^k + 22 = (120+1)11^{2k} + (24+1)25^k + 22$ (OE) $= 120.11^{2k} + 24.25^k + 24\lambda = 24\mu$ $\Rightarrow f(k+1) = 24\mu + 24\lambda = 24(\mu + \lambda) \Rightarrow H_{k+1}$ is true. Hence by PMI H_n is true for all non-negative integers. (Must see non-negative integers.) CSO: Final mark requires all previous marks.	B1 B1 M1 A1 A1 (M1) (A1) (A1) A1 [6]
4	$m^2 - 6m + 25 = 0 \Rightarrow m = 3 \pm 4\mathrm{i}$ CF: $x = e^{3t}(A\cos 4t + B\sin 4t)$ PI: $x - p\sin 2t + q\cos 2t \Rightarrow \dot{x} = 2p\cos 2t - 2q\sin 2t \Rightarrow \ddot{x} = -4p\sin 2t - 4q\cos 2t$ $\Rightarrow 21p + 12q = 195$ $-12p + 21q = 0$ $\Rightarrow p = 7, q = 4$ GS: $x = e^{3t}(A\cos 4t + B\sin 4t) + 7\sin 2t + 4\cos 2t$	M1 A1 M1 M1 A1 A1 [6]

Qn & Part	Solution	Marks
5	Closed curve through $(a, 0)$, $\left(2a, \frac{1}{2}\pi\right)$, (a, π), $\left(0, \frac{3}{2}\pi\right)$. Completely correct shape for cardioid.	B1 B1 [2]
	Area $= \dfrac{a^2}{2}\displaystyle\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}(1+\sin\theta)^2\,\mathrm{d}\theta = \dfrac{a^2}{2}\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}(1+2\sin\theta+\sin^2\theta)\,\mathrm{d}\theta$ (LNR)	M1
	$= \dfrac{a^2}{2}\displaystyle\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}\left(\frac{3}{2}+2\sin\theta-\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$ (LR)	M1
	$= \dfrac{a^2}{2}\left[\dfrac{3}{2}\theta-2\cos\theta-\dfrac{1}{4}\sin 2\theta\right]_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}$	M1
	$= a^2\left(\dfrac{1}{4}\pi+1+\dfrac{1}{8}\sqrt{3}\right)$ (CAO)	A1 [4]
6	$\begin{pmatrix} 2 & -1 & 1 & 3 \\ 2 & 0 & 0 & 5 \\ 6 & -2 & 2 & 11 \\ 10 & -3 & 3 & 19 \end{pmatrix} \to \ldots \to \begin{pmatrix} 2 & -1 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	M1A1
	R(**M**) $= 4 - 2 = 2$ Basis for range space consists of any two independent column vectors from matrix **M**.	A1 A1 [4]
	$2x - y + z + 3t = 0$ $y - z + 2t = 0$	M1
	$t = \lambda, z = \mu, y = \mu - 2\lambda, x = -\frac{5}{2}\lambda$ (OE)	M1
	Basis for null space is $\left\{\begin{pmatrix} -5 \\ -4 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\}$ (OE)	A1A1 [4]

Qn & Part	Solution	Marks
6	**Alternative Solution Using Transpose Matrix**	
	$\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 2 & 2 & 6 & 10 \\ -1 & 0 & -2 & -3 \\ 1 & 0 & 2 & 3 \\ 3 & 5 & 11 & 19 \end{pmatrix} \to \ldots \to \begin{pmatrix} 2 & 2 & 6 & 10 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	M1A1
	R(**M**) = 4 – 2 = 2	A1
	Basis for range space consists of any two independent column vectors from matrix **M**.	A1 [4]
	N.B. The following may appear in the first line, possibly in a 4 × 3 matrix.	
	Using the same set of elementary row operations on the matrix: $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$	
	$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \to \ldots \to \begin{pmatrix} a \\ a+2b \\ b+c \\ -5a-4b+2d \end{pmatrix}$	M1
	Basis for null space is $\left\{ \begin{pmatrix} -5 \\ -4 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ (OE)	M1 A1A1 [4]
7	$(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$	M1
	$= c^5 + 5c^4(\mathrm{i}s) + 10c^3(\mathrm{i}s)^4 + 10c^2(\mathrm{i}s)^3 + 5c(\mathrm{i}s)^4 + (\mathrm{i}s)^5$	A1
	$\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta} = \dfrac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4} = \ldots = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ (AG)	M1A1 [4]
	$\tan 5\theta = 0 \Rightarrow t(t^4 - 10t^2 + 5) = 0$ (Allow non-factorised form.)	M1
	Rejecting $t = 0$, which implies $\theta = k\pi$, roots of $t^4 - 10t^2 + 5 = 0$ are:	A1
	$\pm\tan \frac{1}{5}\pi$ and $\pm\tan \frac{2}{5}\pi$, since $5\theta = \pm 2\pi$ or $5\theta = \pm 4\pi$ (AG)	A1 [3]
	Product of roots $= \tan^2 \frac{1}{5}\pi \tan^2 \frac{2}{5}\pi = 5 \Rightarrow \tan \frac{1}{5}\pi \tan \frac{2}{5}\pi = \sqrt{5}$	M1A1 [2]
	(S.C. Award B1√ if result deduced from information given in the question.)	

Qn & Part	Solution	Marks
8 (i)	$\dot{x} = 2t \quad \dot{y} = 1 - t^2$	B1
	$\dot{x}^2 + \dot{y}^2 = 4t^2 + 1 - 2t^2 + t^4 = (1+t^2)^2$ ACF	M1A1
	$s = \int_0^1 (1+t^2)\,dt = \left[t + \frac{t^3}{3}\right]_0^1 = \frac{4}{3}$ or 1.33 LR	M1A1 [5]
(ii)	$S = 2\pi\int_0^1 \left(t - \frac{1}{3}t^3\right)\left(1+t^2\right)dt = 2\pi\int_0^1 \left(t + \frac{2}{3}t^3 - \frac{1}{3}t^5\right)dt$ LNR	M1A1
	$= 2\pi\left[\frac{1}{2}t^2 + \frac{1}{6}t^4 - \frac{1}{18}t^6\right]_0^1$ LR	M1A1
	$= \frac{11}{9}\pi$ or 3.84	A1 [5]
9	$\mathbf{M}\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix} \Rightarrow$ eigenvalue $= -1 \quad (= \lambda_1)$	M1A1 [2]
	$\lambda_1 + \lambda_2 + \lambda_3 = -8 \Rightarrow \lambda_2 + \lambda_3 = -7$	M1
	$-\lambda_2\lambda_3 = \begin{vmatrix}-2 & 2 & 2\\ 2 & 1 & 2\\ -3 & -6 & -7\end{vmatrix} = -10 + 16 - 18 = -12 \Rightarrow \lambda_2\lambda_3 = 12$	M1A1 A1A1
	$\lambda_2 = -3$ and $\lambda_3 = -4$ or vice versa. (S.C. Award M1A1 for $\lambda^3 + 8\lambda^2 + 19\lambda + 12 = 0$ and A1 for both $\lambda_2 = -3$ and $\lambda_3 = -4$.)	
	$\lambda_2 = -3 \Rightarrow \mathbf{e}_2 = \begin{pmatrix}2\\-1\\0\end{pmatrix}$ (OE)	M1A1
	$\lambda_3 = -4 \Rightarrow \mathbf{e}_3 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ (OE)	A1 [8]

Qn & Part	Solution	Marks
10	$\mathrm{I}_{\mathrm{n}+1} = \int_0^{\frac{1}{4}\pi} \frac{\sin^{2\mathrm{n}+2} x}{\cos x} \mathrm{d}x$ (LR)	B1
	$I_{n+1} = \int_0^{\frac{1}{4}\pi} \frac{(1-\cos^2 x)\sin^{2n} x}{\cos x} \mathrm{d}x$ (LR)	M1
	$= I_n - \int_0^{\frac{1}{4}\pi} \cos x \sin^{2n} x \, \mathrm{d}x$	A1
	$= I_n - \left[\frac{\sin^{2n+1} x}{2n+1}\right]_0^{\frac{1}{4}\pi}$	M1
	$\Rightarrow I_n - 1_{n+1} = \frac{2^{-\left(n+\frac{1}{2}\right)}}{2n+1}$. (AG)	A1 [5]
	$I_0 = \int_0^{\frac{1}{4}\pi} \sec x \, \mathrm{d}x = \left[\ln\lvert\sec x + \tan x\rvert\right]_0^{\frac{1}{4}\pi} = \ln\left(1+\sqrt{2}\right)$	M1A1
	$I_1 = \ln\left(1+\sqrt{2}\right) - \frac{1}{\sqrt{2}}$	M1
	$I_2 = \ln\left(1+\sqrt{2}\right) - \frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}$	A1
	$I_3 = \ln\left(1+\sqrt{2}\right) - \frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}} - \frac{1}{20\sqrt{2}} = \ln\left(1+\sqrt{2}\right) - \frac{73\sqrt{2}}{120}$ (AG)	A1 [5]

Qn & Part	Solution	Marks
11	$\overrightarrow{AB}=\begin{pmatrix}6\\4\\-8\end{pmatrix}$ $\overrightarrow{CD}=\begin{pmatrix}5\\-2\\-4\end{pmatrix}$	B1
	Common perpendicular is $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&2&-4\\5&-2&-4\end{vmatrix}=\begin{pmatrix}-16\\-8\\-16\end{pmatrix}\sim\begin{pmatrix}2\\1\\2\end{pmatrix}$	M1A1
	$\overrightarrow{AD}=\begin{pmatrix}1\\-4\\9\end{pmatrix}\Rightarrow$ shortest distance = $\left\lvert\dfrac{\begin{pmatrix}1\\-4\\9\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix}}{\sqrt{4+1+4}}\right\rvert=\dfrac{16}{3}$ or 5.33	M1A1 [5]
	Normal to Π_1 is $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-4&9\\3&2&-4\end{vmatrix}=\begin{pmatrix}-2\\31\\14\end{pmatrix}$	M1A1
	Normal to Π_2 is $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-4&9\\5&-2&-4\end{vmatrix}=\begin{pmatrix}34\\49\\18\end{pmatrix}$	A1
	$\cos\theta=\dfrac{-2\times34+31\times49+14\times18}{\sqrt{2^2+31^2+14^2}\sqrt{34^2+49^2+18^2}}$	M1A1√
	$\Rightarrow\theta=36.7°$ (CAO)	A1 [6]

Qn & Part	Solution	Marks
12E (i)	$\dot{x} = 2t \quad , \quad \dot{y} = -\frac{1}{2}(2-t)^{-\frac{1}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{4t(2-t)^{\frac{1}{2}}}.$	M1A1
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left\{4t(2-t)^{\frac{1}{2}}\right\}^{-2}\left\{4(2-t)^{\frac{1}{2}} - \frac{4t}{2(2-t)^{\frac{1}{2}}}\right\} \times \frac{1}{2t}$ (OE)	M1A1
	$= \frac{1}{16t^3(2-t)^{\frac{3}{2}}}(4-3t)$ (Any correct simplified form.)	A1 [5]
(ii)	$\int_0^4 y\mathrm{d}x = \int_0^2 2t(2-t)^{\frac{1}{2}}\mathrm{d}t$	
	$= 2\left\{\left[-\frac{2}{3}t(2-t)^{\frac{3}{2}}\right]_0^2 + \int_0^2 \frac{2}{3}(2-t)^{\frac{3}{2}}\mathrm{d}t\right\}$ (LNR)	M1A1
	$= 2\left\{0 + \left[-\frac{4}{15}(2-t)^{\frac{5}{2}}\right]_0^2\right\} = \frac{32}{15}\sqrt{2}$ (LR)	M1A1
	MV $= \frac{\int_0^4 y\mathrm{d}x}{4-0} = \frac{1}{4} \times \frac{32}{15}\sqrt{2} = \frac{8}{15}\sqrt{2}$ or 0.754.	M1A1 [6]
(iii)	$\frac{1}{2}\int_0^4 y^2\mathrm{d}x = \frac{1}{2}\int_0^2 2t(2-t)\mathrm{d}t = \left[t^2 - \frac{1}{3}t^3\right]_0^2 = \frac{4}{3}$ or 1.33.	M1A1
	$\bar{y} = \frac{\frac{1}{2}\int_0^4 y^2\mathrm{d}x}{\int_0^4 y\mathrm{d}x} = \frac{4}{3} \times \frac{15}{32\sqrt{2}} = \frac{5}{8\sqrt{2}} = \frac{5}{16}\sqrt{2}$ or 0.442.	A1 [3]
12O (i)	$d = -2$	B1 [1]
(ii)	$x = 0 \Rightarrow y = -2 = \frac{c}{d} \Rightarrow c = 4$	M1A1
	$y = ax + (b + 2a) + \frac{4 + 4a + 2b}{x - 2}$	M1A1
	Oblique asymptote $y = x + 1 \Rightarrow a = 1, b + 2a = 1 \Rightarrow b = -1$	A1A1 [6]
(iii)	$y = \frac{x^2 - x + 4}{x - 2} \Rightarrow x^2 - (y+1)x + 2y + 4 = 0$	M1
	For real x $B^2 - 4AC > 0 \Rightarrow (y + 1)^2 - 4(2y + 4) > 0$	M1
	$\Rightarrow y^2 - 6y - 15 > 0$	A1
	$\Rightarrow (y - 3)^2 > 24$ (or for solving $y^2 - 6y - 15 = 0$)	M1A1
	$\Rightarrow y - 3 < -2\sqrt{6}$ or $y - 3 > -2\sqrt{6}$ (or for thumbnail sketch)	M1
	$\Rightarrow y < 3 - 2\sqrt{6}$ or $y > 3 + 2\sqrt{6}$ (AG)	A1
	(Note: if done by differentiation, max and min must be demonstrated for full marks.)	[7]