**CAMBRIDGE INTERNATIONAL EXAMINATIONS**

**GCE Advanced Level**

# MARK SCHEME for the October/November 2013 series

# 9231 FURTHER MATHEMATICS

**9231/23** Paper 2, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the October/November 2013 series for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level components and some Ordinary Level components.

**Mark Scheme Notes**

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking $g$ equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

**Penalties**

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

| Question Number | Mark Scheme Details | | | Part Mark | Total |
|---|---|---|---|---|---|
| **1** | Find radial component of acceleration: | $(2 - 2 \times 3 + 3^2)^2 / 0{\cdot}5 = 50$ [m s$^{-2}$] | M1 A1 | | |
| | Find transverse component of acceleration: | $-2 + 2 \times 3 = 4$ [m s$^{-2}$] | B1 | 3 | **3** |
| **2** | Use conservation of momentum, e.g.: | $4mv_A + \lambda mv_B = 4mu$ | B1 | | |
| | Use restitution (must be consistent with prev. eqn.): | $v_A - v_B = -\frac{1}{2} u$ | B1 | | |
| | Solve for $v_B$ : | $4(v_B - \frac{1}{2} u) + \lambda v_B = 4u$ | | | |
| | (or verify eqns are satisfied by this $v_B$) | $v_B = 6u / (\lambda + 4)$ **A.G.** | M1 A1 | 4 | |
| | Use conservation of momentum, e.g.: | $\lambda mw_B + mw_C = \lambda mv_B$ | B1 | | |
| | Use restitution (must be consistent with prev. eqn.): | $w_B - w_C = -\frac{1}{2} v_B$ | B1 | | |
| | Eliminate $w_B$: | $(1 + \lambda)w_C = (1 + \frac{1}{2})\lambda v_B$ | M1 | | |
| | Put $w_C = u$, substitute for $v_B$ and solve for $\lambda$: | $(1 + \lambda) = 9\lambda /(\lambda + 4)$ | | | |
| | | $\lambda^2 - 4\lambda + 4 = 0,\ \lambda = 2$ | M1 A1 | 5 | **9** |

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| **3** | Equate radial forces at $A$ and $B$ (M1 for either): | $T - mg\cos\theta = mv_A^2/a$ | M1 A1 | | |
| | | $T/8 + mg\sin\theta = mv_B^2/a$ | A1 | | |
| | Find 2 energy eqns (M1 for either), e.g.: | $\frac{1}{2}mv_A^2 = \frac{1}{2}mu^2 - mga(1 - \cos\theta)$ | M1 A1 | | |
| | | $\frac{1}{2}mv_B^2 = \frac{1}{2}mu^2 - mga(1 + \sin\theta)$ | A1 | | |
| | Find $u$ by e.g, first eliminating $T$: | $mv_A^2/a + mg\cos\theta =$ | | | |
| | | $8\,mv_B^2/a - 8\,mg\sin\theta$ | | | |
| | | $v_A^2 = 8v_B^2 - 8ga(4/5) - ga(3/5)$ | | | |
| | | $= 8v_B^2 - 7ga$ | M1 | | |
| | and then finding one of $v_A^2$ or $v_B^2$: | $\frac{1}{2}mv_A^2 = \frac{1}{2}mv_B^2$ | | | |
| | | $+ mga(\cos\theta + \sin\theta)$ | | | |
| | | $v_A^2 = v_B^2 + (14/5)\,ga$ | | | |
| | | $v_A^2 = (21/5)ga$ *or* $v_B^2 = (7/5)ga$ | M1 | | |
| | Hence $u$ : | $u^2 = v_A^2 + (4/5)ga$ | | | |
| | | *or* $v_B^2 + (18/5)ga$ | | | |
| | | $= 5ga,\ u = \sqrt{(5ga)}$ | A1 | 9 | **9** |

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| **4** | Resolve vertically at equilibrium with modulus $\lambda$: | $\lambda a / 4a = mg$ [$\lambda = 4mg$] | M1 A1 | | |
| | Use Newton's Law at general point: | $m\, d^2x/dt^2 = mg - \lambda (a+x)/4a$ | | | |
| | | [ *or* $-mg + \lambda (a-x)/4a$ ] | M1 A1 | | |
| | Simplify to give standard SHM eqn: | $d^2x/dt^2 = -(g/a)\, x$ | A1 | | |
| | S.R.: Stating this without derivation (max 4/6): | | (B1) | 6 | |
| | Find period $T = 2\pi/\omega$ with $\omega = \sqrt{(g/a)}$: | $T = 2\pi/\sqrt{(g/a)}$ *or* $2\pi\sqrt{(a/g)}$ | B1 | | |
| | Equate speed at $P$ to one-half maximum speed: | $\omega^2(A^2 - x^2) = ¼\, \omega^2 A^2$, | M1 A1 | | |
| | Find $x^2$: | $x^2 = ¾\, A^2 = ¾\, (½\, a)^2$ [$= 3a^2/16$] | A1 | 4 | |
| | Find $OP$: | $OP = (5 \pm ¼\sqrt{3})\, a$ (A.E.F.) | A1 | | **10** |
| **5** **(i)** | Find $R_P$ by e.g. moments about $Q$ for disc: | $R_P\, r \cos 60° = W\, r \sin 60°$ | M1 A1 | | |
| | | $R_P = W \tan 60° = \sqrt{3}\, W$ **A.G.** | A1 | 3 | |
| **(ii)** | Find $R_Q$ by resolving vertically for disc: | $R_Q \cos 60° = W,\ R_Q = 2W$ | B1 | 3 | |
| | Find $R_B$ by e.g. moments about $A$ for $AB$: | $R_B\, 3a \sin 60°$ | | | |
| | | $= 2W\, (3a/2) \cos 60° + R_Q\, a$ | M1 A1 | | |
| | | $R_B = W(3/2 + 2) / (3\sqrt{3}/2)$ | | | |
| | | $= (7\sqrt{3} / 9)\, W$ **A.G.** | A1 | 4 | |
| | Resolve horizontally and vertically for rod | $X_A = R_B - R_Q \sin 60°$ | M1 | | |
| | (M1 for either) | $= -(2\sqrt{3}/9)\, W$ | A1 | | |
| | (or for rod and disc): | $Y_A = 2W + R_Q \cos 60° = 3W$ | A1 | | |
| | Find magnitude $R$ of reaction at $A$: | $R = \sqrt{(4 \times 3/81 + 9)}\, W$ | M1 | | |
| | | $= \sqrt{(247/27)}\, W = 3{\cdot}02\, W$ | A1 | 5 | **12** |

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| **6** | State or find E($T$): | $E(T) = 1/0{\cdot}2 = 5$ | B1 | 1 | |
| | State or find distribution function of $T$: | $F(t) = 1 - \exp(-0{\cdot}2t)$ ($t$ ù 0) | B1 | | |
| | | = 0 (otherwise *or* $t < 0$) | B1 | | |
| | Find $P(T > 10)$: | $P(T > 10) = 1 - F(10) = 1 - (1 - e^{-2})$ | M1 | | |
| | | $= e^{-2}$ *or* 0·135 | A1 | 4 | **5** |
| **7** | Find $\Sigma (x - \bar{x})^2$ : | $\Sigma (x - \bar{x})^2 = 25 - 10^2/n$ | B1 | | |
| | Find $\Sigma (y - \bar{y})^2$ : | $\Sigma (y - \bar{y})^2 = 43{\cdot}5 - 15^2/2n$ | B1 | | |
| | Equate pooled estimate of $\sigma^2$ to 2: | $(25 - 10^2/n + 43{\cdot}5 - 15^2/2n)$ / <br> $(3n - 2) = 2$ | M1 A1 | | |
| | Rearrange to give quadratic eqn for $n$: | $12n^2 - 145n + 425 = 0$ | M1 | | |
| | Find $n$ (integer value): | $n = (145 \pm 25)/24 = 5$ ($\neq 7{\cdot}08$) | M1 A1 | 7 | **7** |
| **8** | Find value of $p$ for binomial dist.: | mean = 150/100, $p = 1{\cdot}5/6 = ¼$ | M1 A1 | | |
| | State (at least) null hypothesis: | $H_0$: B(6, $p$) fits data (A.E.F.) | B1 | | |
| | Find expected binomial values (to 1 d.p.): | 17·80 35·60 29·66 13·18 3·30 0.44 0.02 | M1 A1 | | |
| | Combine last four cells since exp. value < 5: | $O$: 11 43 35 11 | | | |
| | | $E$: 17·80 35·60 29·66 16·94 | *M1 | | |
| | Calculate value of $\chi^2$ (to 2 d.p.; A1 dep *M1): | $\chi^2 = 2{\cdot}60 + 1{\cdot}54 + 0{\cdot}96 + 2{\cdot}08$ | | | |
| | | = 7·18 (*or* 7·14 if 1 d.p.) | M1 *A1 | | |
| | State or use consistent tabular value (to 2 d.p.): | $\chi_{2,\,0.95}{}^2 = 5{\cdot}991$ (cells combined) | *B1 | | |
| | | $[\chi_{3,\,0.95}{}^2 = 7{\cdot}815,\ \chi_{4,\,0.95}{}^2 = 9{\cdot}488]$ | | | |
| | Correct conclusion (A.E.F., dep *A1, *B1): | $\chi^2 > 5{\cdot}99$ so distn. does not fit | B1 | 10 | **10** |

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| **9** | Calculate sample mean: | $\bar{x} = 94{\cdot}5 / 9 = 10{\cdot}5$ | M1 | | |
| | Estimate population variance: | $s^2 = (993{\cdot}6 - 94{\cdot}5^2/9) / 8$ | | | |
| | (allow biased here: $0{\cdot}15$ *or* $0{\cdot}3873^2$) | $= 0{\cdot}16875$ *or* $0{\cdot}4108^2$ | M1 | | |
| | State hypotheses (A.E.F.): | $H_0$: $\mu = 10{\cdot}2$, $H_1$: $\mu \neq 10{\cdot}2$ | B1 | | |
| | Calculate value of $t$ (to 3 s.f.): | $t = (\bar{x} - 10{\cdot}2)/(s/\sqrt{9}) = 2{\cdot}19$ | M1 *A1 | | |
| | State or use correct tabular $t$ value (to 3 s.f.): | $t_{8,\,0.975} = 2{\cdot}306$ | *B1 | | |
| | (or can compare $\bar{x}$ with $10{\cdot}2 + 0{\cdot}316 = 10{\cdot}52$ ) | | | | |
| | Correct conclusion (AEF, dep *A1, *B1): | Population mean is 10·2 | B1 | 7 | |
| | Find confidence interval<br>(allow $z$ in place of $t$) e.g.: | $10{\cdot}5 \pm t\sqrt{\{1{\cdot}35/(8 \times 9)\}}$ | M1 | | |
| | Use of correct tabular value: | $t_{8,\,0.95} = 1{\cdot}86[0]$ | A1 | | |
| | Evaluate C.I. correct to 3 s.f.: | $10{\cdot}5 \pm 0{\cdot}255$ *or* [10·2, 10·8] | A1 | 3 | **10** |

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| **10** | Find correlation coefficient $r$: | $r = (24{\cdot}25 - 7{\cdot}5 \times 38{\cdot}6/12)$ / $\sqrt{\{(4{\cdot}73 - 7{\cdot}5^2/12)(124{\cdot}84 - 38{\cdot}6^2/12)\}}$ | M1 A1 | | |
| | (A0 if only 3 s.f. used) | $= 0{\cdot}125 / \sqrt{(0{\cdot}0425 \times 0{\cdot}6767)}$ | | | |
| | | $= 0{\cdot}125 / (0{\cdot}2062 \times 0{\cdot}8226)$ | | | |
| | | [*or* $0{\cdot}01042 / \sqrt{(0{\cdot}003542 \times 0{\cdot}05639)}$ | | | |
| | | $= 0{\cdot}01042 / (0{\cdot}05951 \times 0{\cdot}2375)$ ] | | | |
| | | $= 0{\cdot}737$ | *A1 | 3 | |
| | Calculate gradient $b$ in $y - \bar{y} = b(x - \bar{x})$ : | $b = 0{\cdot}125 / 0{\cdot}0425 = 2{\cdot}94[1]$ | B1 | | |
| | Find regression line of $y$ on $x$ (A.E.F.): | $y = 38{\cdot}6/12 + 2{\cdot}94\,(x - 7{\cdot}5/12)$ | M1 | | |
| | (allow use of $x$ on $y$) | $= 3{\cdot}21[7] + 2{\cdot}94\,(x - 0{\cdot}625)$ | | | |
| | | *or* $1{\cdot}38 + 2{\cdot}94x$ | A1 | | |
| | Find $y$ when $x = 0{\cdot}64$: | $y = 3{\cdot}26$ [kg] | B1 | 4 | |
| | State both hypotheses: | $H_0$: $\rho = 0$, $H_1$: $\rho \neq 0$ | B1 | | |
| | State or use correct tabular two-tail $r$ value: | $r_{12,\,2\%} = 0{\cdot}658$ | *B1 | | |
| | Valid method for reaching conclusion: | Reject $H_0$ if $\lvert r\rvert$ > tabular value | M1 | | |
| | Correct conclusion (AEF, dep *A1, *B1): | There is non-zero correlation | A1 | 4 | **11** |

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| **11 (a) (i)** | State or find MI of rod $AB$ about $C$: | $I_{AB} = \frac{1}{3}\, 4m(3a)^2 + 4m(7a)^2$ | M1 A1 | | |
| | | $[\, = 208\, ma^2\, ]$ | | | |
| | Find MI of disc about $C$: | $I_{disc} = \frac{1}{2}\, 8m(2a)^2 + 8m(2a)^2$ | M1 A1 | | |
| | | $[\, = 48ma^2\, ]$ | | | |
| | Find MI of body about $C$: | $I = I_{AB} + I_{disc} = 256ma^2$ | A1 | | |
| | Use eqn of circular motion to find $d^2\theta/dt^2$ : | $I\, d^2\theta/dt^2 = [-]\ (4 \times 7 + 8 \times 2)mga \sin\theta$ | M1 A1 | | |
| | Approximate $\sin\theta$ by $\theta$ and substitute for $I$: | $d^2\theta/dt^2 = -\,(11g/64a)\ \theta$ | A1 | | |
| | Find period $T = 2\pi/\omega$ with $\omega = \sqrt{(11g/64a)}$: | $T = 16\pi\sqrt{(a/11g)}$ **A.G.** | B1 | 9 | |
| **(ii)** | Use energy to find max. ang. vel. $\omega$: | $\frac{1}{2}I\omega^2 = 4mg\ 7a\ (1 - \cos\theta)$ | | | |
| | | $+\ 8mg\ 2a\ (1 - \cos\theta)$ | M1 A1 | | |
| | Substitute for $I$ and $\cos\theta$ and simplify: | $\omega^2 = 2\ (11mga)\ 0{\cdot}4\ /\ (256ma^2)$ | | | |
| | | $= 11g\ /\ 80a$ | A1 | | |
| | Find maximum speed of $A$ (A.E.F.): | $v_A = 10a\omega = \sqrt{(55ga/4)}$ | M1 A1 | 5 | **14** |

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| **11 (b)** | Find F($x$) for $0 < x < 60$ by integration: | $F(x) = ⅓ (x – 20)^3/24000$ $+ ⅓\, 20^3/24000$ | M1 | | |
| | | $= (x – 20)^3 / 72000 + 1/9$ | A1 | | |
| | State F($x$) for other values of $x$: | $F(x) = 0\ (x \le 0),\ 1\ (x \ge 60)$ | B1 | | |
| | Find G($t$) for $0 < t < 60$ from $X + T = 60$: | $G(t) = P(T < t)$ $= P(60 – X < t)$ $= P(X > 60 – t)$ $= 1 – F(60 – t)$ | | | |
| | | $= 8/9 – (40 – t)^3 / 72000$ **A.G.** | M1 A1 | 5 | |
| | Formulate eqn for median $m$ of $T$: | $8/9 – (40 – m)^3 / 72000 = ½$ | M1 | | |
| | Find value of $m$: | $(40 – m)^3 = (8/9 – ½)\ 72000$ $= 28000$ | M1 | | |
| | | $m = 40 – 28000^{1/3} = 9{\cdot}63$ | A1 | | |
| | Find g($t$) for $0 < t < 60$: | $g(t) = (40 – t)^2 / 24000$ | M1 A1 | | |
| | Find E($T$) from $\int t\, g(t)\, dt$: | $E(T) = \int(40^2 t – 80t^2 + t^3)\, dt /24000$ | M1 | | |
| | | $= [½\, 40^2 t^2 – ⅓\, 80t^3 + ¼t^4]_0^{60}/24000$ | A1 | | |
| | | $= 120 – 240 + 135 = 15$ | A1 | 9 | **14** |